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THE MORSE LEMMA ON ARBITRARY BANACH SPACES

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In [4], the author proved the Morse lemma on a real Banach space E which is the dual space of some space E_0, as for example the Sobolev spaces L_p^k, $k \geqq 0$, $1 < p < \infty$ and the Hölder spaces $C^{k,\alpha}$ [5]. The author's first result extended earlier versions by Morse and Palais [2], [3]. In this note we state a theorem and sketch a proof of the Morse lemma for *any* Banach space.

Let $f : U \to R$ be at least C^3 (3 times differentiable) with $0 \in U$ a critical point of f ($Df_0 = 0$). By the Taylor theorem we can write f as

$$f(x) = \tfrac{1}{2}\langle A_x x, x\rangle + f(0)$$

where $A : U \to L(E, E^*)$ {the linear maps from E to E^*} is C^1 and symmetric; i.e.,

$$\langle A_x u, v\rangle = \langle A_x v, u\rangle \qquad \forall u, v \in E.$$

Here $\langle A_x u, v\rangle$ denotes the standard bilinear pairing of E and E^*.

DEFINITION. 0 is said to be a nondegenerate critical point if

(1) $\exists$a nbhd $N \subset U$ of 0 and constants C_1 and C_2 so that $\forall t, t', t_1, t_2 \in N$.

(a) A_t^* is injective (thus A_t is injective).

(b) $\|DA_t(h)(y)\| \leqq C_1\|h\| \cdot \|A_{t'}y\|$ for all $h, y \in E$.

(c) $\|DA_{t_1}(h)(y) - DA_{t_2}(h)(y)\| \leqq C_2\|h\| \cdot \|t_1 - t_2\|\ \|A_{t'}y\|$ for all h, $y \in E$, where D denotes the Fréchet derivative of A with respect to the subscript variable.

(2)(a) For each $t \in N$, $\langle A_t x_n, y\rangle$ converges to zero for all y iff $\langle A_0 x_n, y\rangle$ converges to zero for all y.

(b) Given $t \in N$ if $\langle A_t x_n, y\rangle$ converges to zero for all $y \in E$ then $\langle DA_t(h)(x_n), y\rangle$ converges to zero of all $y \in E$ and $h \in E$.

It is not difficult to check that if $E = H$ (Hilbert space) and $A_0 : H \to H$ is an isomorphism (the standard definition of nondegeneracy) then conditions (1) and (2) are satisfied.

THEOREM (MORSE LEMMA). *Let $f : U \to R$ be C^3 with $0 \in U$ a nondegenerate critical point of f. Then there exists a local diffeomorphism ϕ of a nbdh of 0 so that*

$$f \circ \varphi(x) = \tfrac{1}{2}D^2 f_0(x, x) + f(0),$$

where $D^2 f_0$ is the second derivative of f at 0.

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We shall sketch the principal idea in the proof. Let $t \in N$ where N is given as above. For each $y \in E$ define $f_y \in (\text{Range } A_t)^*$ by $f_y(A_t x) = \langle A_0 x, y\rangle$. Condition 2(a) guarantees that f_y is continuous on Range A_t. Condition 1(a) says that $\overline{\text{Range } A_t} = E^*$. Thus 1(a) and 2(a) together imply that f_y can be extended uniquely to an element of E^{**}.

Let Γ be the set of functionals on E^* induced by E (the weak * topology). Condition 2(a) says that f_y is Γ-continuous for each y and therefore by a well-known result (cf. [1, p. 420]) in functional analysis $f_y \in \Gamma$. Thus there is an element $Q_t y \in E$ so that

$$\langle A_t x, Q_t y\rangle = \langle A_0 x, y\rangle \text{ or } \langle A_t Q_t y, x\rangle = \langle A_0 y, x\rangle \quad \forall x, y.$$

This implies that $A_t Q_t = A_0$. From conditions 1(b) and 1(c) and 2(b) one can show that $Q_t \in GL(E)$ and that $t \to Q_t$ is C^1. Now $Q_0 = I$ and so locally we get a map $P_t \in GL(E), t \to P_t, C^1$ with $P_t^2 = Q_t$. Set $R_t = P_t^{-1}$. As in [**4**] it follows that $\langle A_t u, v\rangle = \langle A_0 R_t u, R_t v\rangle, \forall u, v \in E$. From the inverse function theorem it follows that the map $\psi: x \to R_x(x)$ has a local inverse ϕ and that for

$$\begin{aligned} f(t) &= \tfrac{1}{2}\langle A_t(t), (t)\rangle + f(0) \\ &= \tfrac{1}{2}\langle A_0 R_t(t), R_t(t)\rangle + f(0), \end{aligned}$$

so that

$$\begin{aligned} f \circ \phi(t) &= \tfrac{1}{2}\langle A_0 t, t\rangle + f(0) \\ &= \tfrac{1}{2} D^2 f_0(t, t) + f(0) \end{aligned}$$

which concludes the proof.

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