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TRACE CLASS, WIDTHS AND THE FINITE APPROXIMATION PROPERTY IN BANACH SPACE

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I. **Introduction.** Although there have been a number of attempts [**3**] to define operator classes in Banach space whose properties are analogous to the classical trace class operators of Hilbert space [**1**], [**2**] it is generally agreed that a satisfactory definition has yet to be achieved [**3**]. The purpose of the present note is to introduce a new approach to the problem wherein operator widths [**2**], [**4**] in Banach space replace the eigenvalues of the Hilbert space formulation; the viability of the approach being illustrated by the formulation of a number of sufficient conditions for an operator to have the finite approximation property in terms of its widths. Moreover, unlike the previous approaches [**3**] the trace class operators defined via operator widths are representation independent and coincide exactly with the classical definitions in Hilbert space.

II. **Definitions and results.** In the sequel X is a Banach space normed by $\|\cdot\|$, B is the unit ball in X and $\mathscr{L}_n$ is the set of n-dimensional subspaces of B. The nth width, $d_n(A)$, of an operator A on X is defined [**4**] by

$$d_n(A) \equiv \inf_{L \in \mathscr{L}_n} \sup_{u \in B} \inf_{v \in L} \|Au - v\|. \tag{1}$$

Classically, Kolmogorov [**4**] defined the nth width of a set to be a measure of the degree to which the set could be approximated by n-dimensional subspaces, the definition of equation (1) being that of Kolmogorov applied to $A(B)$.

Some remarks concerning the sequence $\{d_n(A)\}$ are as follows:

(a) $d_0(A) = \|A\|$;

(b) $\{d_n(A)\}$ is a nonincreasing sequence and $d_n(A) \to 0$ iff A is a compact operator;

(c) (see for example [**2**]) for X a Hilbert space and A a compact linear operator on X, set $s_n(A) \equiv \lambda_n((A^*A)^{1/2}) \equiv$ the nth eigenvalue of $(A^*A)^{1/2}$ $(n = 1, 2, \ldots)$ (these are called the s numbers or characteristic numbers of A). Then $d_n(A) = s_{n+1}(A)$ $(n = 0, 1, 2, \ldots)$. A is an Hilbert-Schmidt or nuclear operator if the sequence $\{s_n(A)\}$ is an l_2 or l_1 sequence.

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We define the "trace" classes D_p of compact linear operators on a Banach space as

$$D_p \equiv \left\{ A \text{ compact, linear} \,\middle|\, \sum_{n=0}^{\infty} d_n(A)^p < \infty \right\}.$$

(In the case of a Hilbert space they reduce to the classes C_p or S_p defined in [**1**] or [**2**].)

Define $\|\cdot\|_p : D_p \to \boldsymbol{R}$ by

$$\|A\|_p \equiv \left[\sum_0^{\infty} d_n(A)^p \right]^{1/p}.$$

THEOREM 1. *$\|A\|_p$ is a norm on D_p.*

An operator $A \in D_1$ will be called d-nuclear.

Before proceeding to our main theorem on the approximation of trace class operators, we introduce another sequence, $p_n(X)$, of positive numbers —which characterize the Banach space X rather than the operator A— defined by

$$p_n(X) \equiv \sup_{L \in \mathscr{L}_n} \inf_{P(L) \in P_L} \|P(L)\|,$$

where $\mathscr{L}_n$ denotes the set of all n-dimensional linear subspaces of X, and P_L denotes the set of all projections into the subspace L.

(i) $1 \leqq p_n(X) \leqq n$.

(ii) If X is a Hilbert space, $p_n(X) \equiv 1$ $(\forall n)$.

(iii) $p_n(X) = p_n(X^*)$ if X is reflective.

(iv) Murray [**5**] has shown that $p_n(X)$ grows linearly with n for $X = L_p$ or l_p $(p \neq 2)$.

We say that A has the finite approximation property iff $\exists$ a sequence of operators, A_n, with finite-dimensional ranges, such that $\|A - A_n\| \to 0$ as $n \to \infty$.

THEOREM 2. *Let A be a compact operator on X such that*

$$\operatorname*{Lim}_{n \to \infty} [d_n(A) p_n(X)] = 0.$$

Then A has the finite approximation property.

Since, in Hilbert space $p_n(X) = 1$, and $d_n(A) \to 0$ for A compact, we trivially obtain the classical result

COROLLARY 3. *Every compact operator on a Hilbert space has the finite approximation property.*

Although we are not able to answer the classical conjecture on the finite approximation property for any compact operator on a Banach space we can observe

COROLLARY 4. *Every $A \in D_1$ has the finite approximation property.*

This holds because $\{d_n(A)\} \in l_1$, hence $d_n(A) = O(1/n^{1+\varepsilon})$ ($\varepsilon > 0$) and $d_n(A)p_n(X) = O(n \cdot 1/n^{1+\varepsilon}) = O(n^{-\varepsilon})$ ($n \to \infty$).

The standard examples of nuclear operators in Hilbert space have natural analogs in Banach space to which the corollary applies. In particular, the injection maps of Sobolev spaces $W_0^{s,p}$ into L^p is d-nuclear, and integral operators with smooth kernels on L^p into L^p are d-nuclear.

PROOF OF THEOREM 2.

$$d_n(A) = \min_{L \in \mathscr{L}_n} \max_{\|x\|=1} \|Ax - L\| = \min_{L \in \mathscr{L}_n} \max_{\|x\|=1} \max_{x^* \in L^\perp; |x^*|=1} \|x^*Ax\|$$

$$= \min_{L \in \mathscr{L}_n} \max_{x^* \in L^\perp; |x^*|=1} \|A^*x^*\|, \quad \text{where } X^* \supset L^\perp \text{ is the annihilator of } L.$$

LEMMA. $P(L)^* = 1 - P(L^\perp)$.

Let $\tilde{L} \in \mathscr{L}_n$ be the minimizing subspace for $d_n(A)$ (which exists since A is compact). Then $\exists$ a projection $P_n = P(\tilde{L})$ such $\|P_n\| = O(p_n(X))$. Set $P_n^\perp \equiv P(\tilde{L}^\perp)$.

We then have

$$d_n(A) = \sup_{x^* \in X^*} \frac{\|A^*P_n^\perp x^*\|}{\|P_n^\perp x^*\|} \geqq \sup_{x^* \in X^*} \frac{\|A^*P_n^\perp x^*\|}{\|P_n^\perp\| \|x^*\|}.$$

Since $\|P_n^\perp\| = \|1 - P_n\| \leqq 1 + \|P_n\|$, we have

$$2p_n(X)d_n(A) \geqq \sup_{x^*} \|A^*P_n^\perp x^*\|/\|x^*\| = \|A^*P_n^\perp\| = \|(1 - P_n)A\|.$$

Setting $A_n \equiv P_nA$, $\|A - A_n\| = O(d_n(A)p_n(X))$ and the theorem is proved.

We conjecture that the condition of Theorem 2 is also necessary.

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