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SINGULARITY SUBSCHEMES AND GENERIC PROJECTIONS

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Let k be an algebraically closed field, and let P^n be projective n-space over k. Let $V^r \subset P^n$ be a smooth projective variety. It is known that V can be embedded in P^{2r+1} (cf. [2]), but there are smooth r-dimensional varieties which cannot be embedded in P^{2r}.

Let $r \leqq m \leqq \min(2r, n-1)$, and let $\pi: V \to P^m$ be induced by projection from an $(n-m-1)$-subspace $L \subset P^m$ with $L \cap V = \emptyset$. As in [4], we ask what can be said about π when L is chosen *generically*, i.e. L is chosen from some dense open subset of the corresponding Grassmann variety. In the case that $m \geqq r+1$, the problem is to describe the local nature of the singular locus of $V' = \pi(V)$. This is of interest because V' can be chosen to be birational to V. Specifically, we would like to describe the structure of the local rings $\hat{\mathcal{O}}_{V',y}$ for closed points $y \in V'$.

For $i > 0$, let $S_i \subset V$ consist of all points x at which the tangent map has rank $\leqq r - i$. Thus

$$S_i = \{x \in V \mid \dim_{k(x)}(\Omega^1_{X/P_m}(x)) \geqq i\}.$$

The following result is known; cf. [5, Lemma 3].

PROPOSITION. *If L is chosen generically, then S_i is of pure codimension $i(m-r+i)$ in V, for all $i > 0$.*

In particular, if $m = r+1$, then $\operatorname{codim}(S_1) = 2$, and $\operatorname{codim}(S_2) = 6$. This says that $S_2 = \emptyset$ if $r \leqq 5$ and $m \geqq r+1$.

Let x be a closed point of $S_1 - S_2$, and let $y = \pi(x)$. Let $\pi^*: \mathcal{O}_{P^m,y} \to \mathcal{O}_{V,x}$ be the corresponding homomorphism of local rings. We can choose parameters $t_1, \ldots, t_r$ (resp. $u_1, \ldots, u_m$) in $\mathcal{O}_{V,x}$ (resp. $\mathcal{O}_{P^m,y}$) such that $\pi^*(u_i) = t_i$ for $i = 1, \ldots, r-1$, while $\pi^*(u_i) \in \mathfrak{m}_x^2$ for $i = r, \ldots, m$, where $\mathfrak{m}_x \subset \mathcal{O}_{V,x}$ is the maximal ideal. In a natural way, one can define closed subschemes $S_1^{(q)} \subset V - S_2$ such that if $\operatorname{char}(k) = 0$, the local generators of the sheaf of ideals defining $S_1^{(q)}$ are $(\partial^j u_i/\partial t_r^j)$ for $1 \leqq j \leqq q$, and $r \leqq i \leqq m$. In general, there are differential operators $D^{(j)}: \mathcal{O}_{V,x} \to \mathcal{O}_{V,x}$ such that $(D^{(j)}f)(x)$ is the coefficient of t_r^j in the power series expansion of f (cf. [1, §16]). The elements $D^{(j)}(\pi^* u_i)$ are the correct local generators.

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THEOREM 1. *Every smooth projective variety V^r has an embedding such that if $\pi : V^r \to P^m$ is induced by generic projection, then*

(i) *$S_1^{(q)}$ is of pure codimension $q(m - r + 1)$ in V, for all q;*

(ii) *if $\operatorname{char}(k) \nmid (q + 1)$ (resp. $\operatorname{char}(k) \mid (q + 1)$), then $S_1^{(q)}$ is smooth over k (resp. fails to be smooth over k at only finitely many points).*

The following result, which is essentially a statement about homomorphisms of formal power series rings, shows how to use Theorem 1 to obtain canonical forms for the homomorphisms $\pi^* : \hat{\mathcal{O}}_{P^m,y} \to \hat{\mathcal{O}}_{V,x}$, when $x \notin S_2$.

THEOREM 2. *Let $\pi : V^r \to W^m$ be a morphism of smooth varieties over k, with $m \geqq r$. Let $x \in V$ be a closed point, let $y = \pi(x)$, and assume that $x \in S_1^{(q)} - S_1^{(q+1)}$, and that $S_1^{(q)}$ is smooth at x. Then the local rings $\hat{\mathcal{O}}_{V,x}$ and $\hat{\mathcal{O}}_{W,y}$ can be identified with formal power series rings $k[[t_1, \ldots, t_r]]$ and $k[[u_1, \ldots, u_m]]$ so that*

(i) *$\pi^*(u_i) = t_i$, for $i = 1, \ldots, r - 1$;*

(ii) *$\pi^*(u_r) = \sum_{j=1}^{\beta} t_{q(m-r)+j} t_r^j + t_r^{q+1}$, where $\beta = q - 1$ (resp. q) if $\operatorname{char}(k) \nmid (q + 1)$ (resp. $\operatorname{char}(k) \mid (q + 1)$);*

(iii) *$\pi^*(u_{r+i}) = \sum_{j=1}^{q} t_{q(i-1)+j} t_r^j$, for $i = 1, \ldots, m - r$.*

REMARK. If $\operatorname{char}(k) \mid (q + 1)$ and $q(m - r + 1) = r$, then $S_1^{(q)}$ is finite and has no smooth points. If $\operatorname{char}(k) = 2$, pinch points of surfaces in P^3 are an example of this; we have $r = 2$, $m = 3$, and $q = 1$ in this case.

The above results give information about the structure of $\hat{\mathcal{O}}_{V',y}/\mathfrak{p}$, where $\mathfrak{p}$ is a minimal prime ideal corresponding to a point $x \in \pi^{-1}(y)$ which satisfies the hypotheses of Theorem 2. The next theorem can be used to show how these minimal prime ideals relate to each other.

Let $2 \leqq a \leqq m + 1$, and suppose that any a points of V span an $(a - 1)$-subspace of P^n. Let $U_a \subset V \times \cdots \times V$ (a copies) consist of all a-tuples of distinct points. We can define a morphism $\phi : U_a \to G(n, a - 1) = \operatorname{Grass}_a(k^{n+1})$, such that $\phi(x_1, \ldots, x_a)$ is the point which corresponds to the subspace Λ spanned by $x_1, \ldots, x_a$. For a fixed $(n - m - 1)$-subspace $L \subset P^n$, let $\Sigma \subset G(n, a - 1)$ be the special Schubert cycle $\Sigma = \{\Lambda \mid \dim(L \cap \Lambda) \geqq a - 2\}$. Thus $(x_1, \ldots, x_a) \in \phi^{-1}(\Sigma)$ iff $\pi(x_1) = \ldots = \pi(x_a)$.

THEOREM 3. *Every smooth projective variety has an embedding such that if L is choosen generically, then $\phi^{-1}(\Sigma)$ is smooth and of codimension $(a - 1)m$ in U_a. Moreover, if $a \geqq 2$, and if $q_1, \ldots, q_a$ are $\geqq 0$, then $\phi^{-1}(\Sigma) \cap (S_1^{(q_1)} \times \cdots \times S_1^{(q_a)})$ is smooth. (We set $S_1^{(0)} = V$.) If $\operatorname{char}(k) \mid (q + 1)$ and $S_1^{(q)}$ is not smooth at x, then $\pi^{-1}(\pi(x)) = \{x\}$, provided that $m \geqq r + 1$.*

The fact that $\operatorname{codim}(\phi^{-1}(\Sigma)) = (a - 1)m$ was proved by E. Lluis [3]; the smoothness of $\phi^{-1}(\Sigma)$ is equivalent to Theorem 1 of [4]. The proof of the last statement as well as the proofs of Theorems 1 and 2 will be published elsewhere.

We will now indicate how Theorem 3 is applied. Let $y \in \boldsymbol{P}^m$, with $\{x_1, \ldots, x_a\} \subset \pi^{-1}(y)$; assume that $x_j \in S_1^{(q_j)}$ for $j = 1, \ldots, a$. For each j, we choose elements $u_{j1}, \ldots, u_{jd_j}$ which form a subset of a regular system of parameters at y such that each $\pi^*(u_{j\nu})$ induces an element of the square of the maximal ideal along $S_1^{(q_j)}$ at x_j. [If $x_j \notin S_1^{(q_j+1)}$, and $\operatorname{char}(k) \nmid (q_j + 1)$, we can take $d_j = q_j(m - r + 1) + (m - r)$.] The smoothness implies that $\{u_{j\nu} \mid 1 \leqq j \leqq a, \text{ and } 1 \leqq \nu \leqq d_j\}$ is a subset of a regular system of parameters at y.

In particular, let us consider the case $r = 3$, $m = 4$, with $\operatorname{char}(k) \neq 2$. In this case, there are finitely many $y \in V'$ with $\pi^{-1}(y) = \{x_1, x_2\}$, where $x_1 \notin S_1$, and $x_2 \in S_1^{(1)} - S_1^{(2)}$. For such a point, the above considerations imply that

$$\mathcal{O}_{V',y} \cong k[[u_1, u_2, u_3, u_4]]/(u_1(u_2^2 - u_3^2 u_4)).$$

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