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OPERATORS ON FUNCTION SPACES

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1. **Introduction.** In this announcement we present characterizations of weakly compact and compact operators defined on function spaces. Besides the space of totally measurable functions, we consider the space of all Banach-valued continuous functions, where the topology of the space is either the compact-open topology or the topology generated by the supremum norm on functions vanishing at infinity. The main tools are a recent result of Brooks [5] concerning weak compactness of vector measures, and integral representation theorems in a very general setting [8] which serve to unify the existing theorems of this type and facilitate the study of operator theory. Our characterization provides a natural and simple condition for operators to be weakly compact—namely that $\tilde{m}(A_i) \to 0$, whenever $A_i \searrow \emptyset$, where $\tilde{m}$ is the semivariation of the representing measure for the operator. This extends the Bartle-Dunford-Schwartz theory [2] for weakly compact operators from $C(S)$ into X. The necessity part of Theorem 1 extends the work of Batt and Berg [4]. Also we give a necessary and sufficient condition, in terms of the underlying topology of the domain space, in order that the classes of weakly compact and compact operators coincide. Finally in §4 we briefly mention additional results concerning operators. In a later paper [7], representations in the setting of locally convex spaces and applications will be given.

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2. **Definitions and notation.** H is a Hausdorff space such that the set of all continuous scalar functions separates points of H. E and F are Banach spaces with conjugate spaces E^* and F^* respectively; E_1^* and F_1^* denote the unit spheres. We regard F as a subset of F^{**}. $C(H, E)$ is the space of all continuous E-valued functions defined on H, where $C(H, E)$ is equipped with the compact-open topology. If H is a locally compact Hausdorff space, $C_0(H, E)$ denotes the space of continuous E-valued functions vanishing at infinity.

$B(E, F^{**})$ is the space of operators (= bounded linear mappings) from E into F^{**}. The σ-algebra of Borel subsets of H is denoted by Σ. If $\boldsymbol{m}: \Sigma \to B(E, F^{**})$ is finitely additive, define the semivariation $\tilde{\boldsymbol{m}}$ of $\boldsymbol{m}$ as follows: $\tilde{\boldsymbol{m}}(A) = \sup|\sum \boldsymbol{m}(A_i)x_i|$, where the supremum is taken over all finite disjoint subsets A_i of A and $x_i \in E_1$. For $z \in F^*$, let $\boldsymbol{m}_z: \Sigma \to E^*$ be defined by $\boldsymbol{m}_z(A)x = \langle \boldsymbol{m}(A)x, z\rangle$, $x \in E$. The total variation function of a set function μ is denoted by $|\mu|$. A *representing measure* $\boldsymbol{m}$ for the operator T defined on either $C(H, E)$ or $C_0(H, E)$ into F is a finitely additive set function $\boldsymbol{m}: \Sigma \to B(E, F^{**})$ with finite semivariation such that: (i) $T(f) = \int f\, d\boldsymbol{m}$ for each f in the domain of T; (ii) $|\boldsymbol{m}_z|$ is a regular Borel measure on Σ, for each $z \in F^*$. We write $T \leftrightarrow \boldsymbol{m}$ to indicate this correspondence. A set function $\boldsymbol{m}$ is *strongly bounded* (s-bounded) if $\tilde{\boldsymbol{m}}(A_i) \to 0$, whenever $\{A_i\}$ is a disjoint sequence of sets. When $T \leftrightarrow \boldsymbol{m}$, this condition is equivalent to $\tilde{\boldsymbol{m}}(A_i) \to 0$, whenever $A_i \searrow \emptyset$. This concept was introduced in Lewis [12] under the name variational semiregularity (vsr). We remark that countable additivity of the measure does not imply s-boundedness [13].

Let $\mathscr{R}$ be a ring of sets. The Banach space of all finitely additive set functions from $\mathscr{R}$ into F, with the total variation norm, is denoted by $fa(\mathscr{R}, F)$. The normed space of totally $\mathscr{R}$-measurable E-valued functions is denoted by $M_E(\mathscr{R})$, where the norm is the supremum norm. Recall that a function is totally $\mathscr{R}$-measurable if it vanishes outside of a set in $\mathscr{R}$ and is the uniform limit of $\mathscr{R}$-measurable simple functions. We say $\boldsymbol{m}$ is a *representing measure* for an operator $T: M_E(\mathscr{R}) \to F$ if $\boldsymbol{m}: \mathscr{R} \to B(E, F)$ is finitely additive with finite semivariation and $T(f) = \int f\, d\boldsymbol{m}$. The bilinear integration theory used is defined in [8].

3. **The main results.**

Theorems 1 and 2 below can be strengthened by assuming that E^* and E^{**} have the Radon-Nikodym property and each $\boldsymbol{m}(A)$ is weakly compact, but for simplicity we impose the stronger assumption that E be reflexive. A similar observation holds for Proposition 3 except that K acting on each A must be relatively weakly compact.

THEOREM 1. *Suppose that T is an operator defined on $C(H, E)$ or $C_0(H, E)$*

into F with representing measure $\boldsymbol{m}$. If T is weakly compact, then $\boldsymbol{m}$ is strongly bounded. Conversely, if E is reflexive and $\boldsymbol{m}$ is strongly bounded, then T is weakly compact.

REMARK 1. If E is not reflexive, then there always exists a nonweakly compact operator with an s-bounded representing measure.

REMARK 2. The above theorem extends the Bartle-Dunford-Schwartz theorem [2] in the case E is the scalar field; in this case countable additivity is equivalent to s-boundedness.

THEOREM 2. *Let $\mathscr{R}$ be a ring of sets. Suppose that $T: M_E(\mathscr{R}) \to F$ is an operator with representing measure $\boldsymbol{m}$. If T is weakly compact, then $\boldsymbol{m}$ is strongly bounded. Conversely, if E is reflexive and $\boldsymbol{m}$ is strongly bounded, then T is weakly compact.*

The proofs of the above theorems use the following two results.

PROPOSITION 3 (BROOKS [5]). *If K is a relatively weakly compact subset of $fa(\mathscr{R}, F)$, then:* (i) *K is bounded;* (ii) *the family $\{|\mu| : \mu \in K\}$ is uniformly strongly additive, that is $|\mu|(A_i) \to 0$ uniformly for $\mu \in K$, whenever $\{A_i\}$ is a disjoint sequence. Conversely, if F is reflexive, conditions* (i) *and* (ii) *imply that K is relatively weakly compact.*

The next proposition uses the techniques in Brooks and Lewis [7]. For related representation theorems see [8], [10], [11] and [18].

PROPOSITION 4. *If T is an operator defined on $C(H, E)$ or $C_0(H, E)$ into F, then T has a unique representing measure.*

REMARK 3. We use the fact that E-valued simple functions can be embedded in $C_0(H, E)^{**}$. In fact, the mapping from $M_E(\Sigma)$ into $C_0(H, E)^{**}$ is an isometric isomorphism. If $C(H, E)$ is the domain of T, then $\boldsymbol{m}$ has compact support.

Recall that a dispersed topological space is a space containing no nonempty perfect sets. $C(H)$ is the space of all continuous scalar-valued functions defined on H. While the equivalence of (a) and (b) in the following theorem was established in [17], we present a more complete statement in terms of vector measures.

THEOREM 5. *Let H be a dispersed compact Hausdorff space and let F be a real Banach space. Suppose that $T: C(H) \to F$ is an operator and $\boldsymbol{m}$ is the representing measure for T. Then the following are equivalent:*

(a) *T is compact;*

(b) *T is weakly compact;*

(c) *$\boldsymbol{m}$ is strongly bounded;*

(d) *$\boldsymbol{m}: \Sigma \to F$.*

Conversely, if conditions (a) *and* (b) *are always equivalent, then the compact*

Hausdorff space H is dispersed.

4. **Additional results.** I. Suppose E is a weakly sequentially complete space in which weak and strong sequential convergence are equivalent. Then every operator $T: C_0(H, E) \to F$, which has an s-bounded representing measure, maps weakly convergent sequences into norm convergent sequences—that is, $C_0(H, E)$ has the strong Dunford-Pettis property. In particular, if $F = C_0(H, E)$ and T is weakly compact, then T^2 is compact. Therefore, no infinite dimensional reflexive subspace of $C_0(H, E)$ can be complemented in $C_0(H, E)$. This extends a result of Grothendieck.

II. It is of interest to determine when $\boldsymbol{m}$ takes its values in the subspace $B(E, F)$ of $B(E, F^{**})$. If $T \leftrightarrow \boldsymbol{m}$, where T is an operator on $C_0(H, E)$, then $\boldsymbol{m}$ takes its values in $B(E, F)$ if and only if $T_x: C_0(H) \to F$ is weakly compact for each $x \in E$ (here $T_x(f) = T(xf)$). It follows that if $\boldsymbol{m}$ is countably additive, then $\boldsymbol{m}$ takes its values in $B(E, F)$.

III. It follows from a result of Pelczyński [16] and Theorem 5 that if F has no subspace isomorphic to c_0 and H is a dispersed compact Hausdorff space, then every operator $T: C(H) \to F$ is compact (cf. [3, Corollary 2, p. 913] and [9, p. 515]).

IV. It was shown in [6] that if E is the scalar field, then the pointwise limit of finitely additive s-bounded measures is s-bounded. This result fails in general; the limit measure can even be chosen to be a countably additive Baire measure. The following are equivalent: (i) the Banach space F does not contain c_0; (ii) for each H and each E the limit of every uniformly bounded (in semivariation) pointwise convergent sequence of s-bounded representing measures m_i: $\Sigma(H) \to B(E, F)$ is s-bounded; (iii) for each H and each E a representing measure $\boldsymbol{m}$: $\Sigma(H) \to B(E, F)$ is countably additive if and only if $\boldsymbol{m}$ is s-bounded.

V. Suppose E is reflexive and F does not contain c_0. If the weakly compact operators $T_n: M_E(\mathscr{R}) \to F$ converge pointwise to T and uniformly on sets $\{x\, \xi_A : x \in E_1\}$ for each $A \in \mathscr{R}$, then T is weakly compact.

VI. The seminorm $\rho(z) = |\boldsymbol{m}_z|(H)$ defined on F^* has been studied in [14]. We show that if T is an operator on $C(H, E)$ and $T \leftrightarrow \boldsymbol{m}$, then T is compact if and only if (F_1^*, ρ) is a compact space; T is compact with dense range if and only if ρ induces the weak* topology on F_1^*.

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