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AN INVERSE PROBLEM FOR GAUSSIAN PROCESSES[1]

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Let $X(t)$ be a centered stationary Gaussian process (c.s.G.p.). Its statistics are completely determined by its correlation function

$$R(s) = E(X(t)X(t + s)).$$

This is a positive definite function and we assume it is continuous at the origin.

A problem often considered in the electrical engineering literature is that of determining the statistics of

$$Y(t) = X^2(t).$$

The best results, to our knowledge, consist of the computation of some few moments of higher order [**1**].

Two problems are considered in this note.

1. Does there exist a universal constant m such that the moments of order $\leqq m$ of $Y(t)$ are enough to determine its statistics? (Recall that $m = 2$ for a Gaussian process.)

2. How much of the statistics of $X(t)$ can you read off from those of $Y(t)$?

The answers to 1 and 2 are embodied in the next two statements.

THEOREM I. *Let m be an arbitrary positive integer. There exists a centered stationary Gaussian process $X(t)$ such that the moments of order $\leqq m$ of $Y(t) = X^2(t)$ do not suffice to determine Y's statistics.*

THEOREM II. *The statistics of $Y(t) = X^2(t)$ determine uniquely those of $X(t)$.*

The proof of this second result appears in [**2**]. Stationarity can be disposed of, but the Gaussian character of the process is essential. Finally the real line as a parameter space can be replaced by any arcwise connected space.

PROOF OF THEOREM I. A simple computation shows that knowing all the moments of order $\leqq m$ of $Y(t)$ is equivalent to knowing the expressions

$$(1) \qquad \sum_{\pi} R(t_{\pi 1} - t_{\pi n})R(t_{\pi 2} - t_{\pi 1}) \ldots R(t_{\pi n} - t_{\pi(n-1)}), \qquad 2 \leqq n \leqq m.$$

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Here $t_1 \ldots t_n$ are arbitrary real numbers, and the sum ranges over the group of all permutations π of n elements.

Consider now the function

$$f(x) = x^2(1 - \cos x)(1 + \varepsilon \cos kx), \qquad |\varepsilon| < 1, k \geqq 2.$$

Its Fourier transform is the positive definite function

$$\begin{aligned} R(s) &= \tfrac{1}{2}\varepsilon(1 - |s - k|) && \text{for } |s - k| \leqq 1, \\ &= 1 - |s| && \text{for } |s| \leqq 1, \\ &= \tfrac{1}{2}\varepsilon(1 - |s + k|) && \text{for } |s + k| \leqq 1, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Take $X(t)$ to be a c.s.G.p. with this correlation function. We claim that, by choosing k appropriately, we can conclude that the information contained in (1) is not enough to determine the sign of ε. Indeed, the only arrangements of t's that give a nonzero contribution to (1) are those for which all the differences $|t_{\pi i} - t_{\pi(i-1)}|$ are either smaller than 2 or else between $k - 1$ and $k + 1$. This plus the fact that

$$(t_{\pi 1} - t_{\pi n}) + \cdots + (t_{\pi n} - t_{\pi(n-1)}) = 0$$

implies that for $k > 2m \geqq 2n$ the number of terms in this sum which are close to k has to match the number of terms which are close to $-k$. But going to the corresponding term in (1) this means that ε enters with an even power and its sign is lost.

Briefly, for any given m we construct a c.s.G.p. $X(t)$ such that the m-order statistics of $Y(t) = X^2(t)$ do not determine the correlation of $X(t)$. The proof is now finished if one invokes Theorem II.

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