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SET VALUED TRANSFORMATIONS

BY D. G. BOURGIN

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The present note summarizes some results in a new algebraic topological approach to set valued transformations and initiates a theory of their fixed points applicable to the case that the images are not acyclic. These ideas extend to min max theorems where again a basic generalization is obtained [2]. Our developments are based on the existence of homomorphisms of certain homology groups, in a crucial range only, induced by suitably defined multivalued homotopies (cf. Theorems 1 and 3 below).

Let X and Y be paracompact spaces and suppose $h:X \times I \to Y$ is a set valued uppersemicontinuous (usc) transformation. Let $\Gamma(h)$ be the graph

$$\Gamma(h) = \bigcup \{(x, s, y) | y \in h(x, s)\} \subset X \times I \times Y.$$

Let p_1 be the projection of $\Gamma(h)$ onto X, p_2 the projection onto Y and P_1 the projection of $\Gamma(h)$ onto $X \times I$.

For each $s \in I$ the *singular set* $S^p(s)$ is defined by

$$S^p(s) = \{x | H^r h(x, s) \not\approx 0 \quad \text{for some } r < p\}$$

where H^* refers to Alexander reduced cohomology over the coefficient field Q and closed support family. Write

$$S^p = \bigcup_{s \in I} S^p(s) \ .$$

We say p_1 is *almost p solid*, ApS, if for any neighborhood $N(y_0)$ in a suitable neighborhood base at $y_0 \in Y$, there is at most a finite subset of S^p, independent of s, such that $h(x, s) \cap N(y_0) \neq \emptyset$, $x \in S^p$, does not imply $h(x, s) \in N(y_0)$ and $h(x, s)$ is uniformly usc for fixed x.

We write $f \sim_{pq} g$ if h_s, $s \in I$, is acyclic for $p \leqq m \leqq q \leqq \infty$ and $p_1(h)$ is ApS. The basic theorem for our purpose is

THEOREM 1. *If $f \sim_{pq} g$, $q \geqq p + 2$ and h describes the homotopy then $h(m)^*: H^m(Y) \to H^m(X \times I)$ exists for $p + 2 \leqq m \leqq q$ and $f^*(m) = g^*(m)$ for this range of m values.*

If X and Y are compacta, a condition designated by (C) is

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S^p is denumerable and for arbitrary positive ε, there is at most a finite subset of S^p for which diam $f(x) > \varepsilon$. As a consequence of Theorem 1 restricted to compact spaces there results

THEOREM 2. *Let f be an uppersemicontinuous set valued self transformation of the $n + 1$ disk, $D^{n+1}, n > 3$. Let S^{n-2} be the singular subset defined by the condition that $f(x)$ is a convex set for $x \bar{\in} S^{n-2}$ and for $x \in S^{n-2}$, $f(x)$ is a finite union of convex sets of which at most $n - 1$ are of dimension greater than $n - 3$. Require also that* (C) *be satisfied. Then f has a fixed point.*

Another type of homotopy theorem is also available. Thus for a set valued transformation f on X to Y we grade the singular set by

$$\mu_r = \{x | H^r f(x) \not\approx 0\}.$$

Let $d_r = \dim \mu_r$ the maximum covering dimension of sets A closed in Y and contained in μ_r. Similarly for h,

$$V_r = \{(x, s) | H^r h(x, s) \not\approx 0\}.$$

Let $\delta_r = \dim V_r$ be the maximum covering dimension of sets A closed in $X \times I$ and contained in V_r. Let

$$\Pi = 1 + \sup_{V_r \neq \emptyset; r < q} (r + \delta_r).$$

The notation $f \sim_{p\Pi q} g$ is used if there is a usc transformation h, said to describe the homotopy with $V_r = \emptyset$, $p \leqq r \leqq q$ and $p \leqq \Pi < q$.

The correspondent to Theorem 1 is

THEOREM 3. *If $f \sim_{p\Pi q} g$ and h describes this homotopy with f, g and h usc then if $q \geqq \Pi + 2$, $h^*(m)$ exists and $f^*(m) = g^*(m)$ for $\Pi + 1 \leqq m < q$.*

For the case that $\delta_r \equiv 0$ for all $r < p$, Theorem 3 includes Theorem 1 if the spaces are *compacta*. (However this is not true if either the compactness or the metrizability restrictions are dropped as can be shown by suitable examples.) Accordingly instead of Theorem 2 we can assert

THEOREM 4. *Let f be a usc transformation on D^{n+1} to D^{n+1}. Let S be the singular set $\bigcup V_r$ with* $\dim S = d$. *Suppose $\mu_r = \emptyset$ for $r \geqq n - 3 - d = \bar{r}$. For $x \bar{\in} S, f(x)$ is convex. For $x \in S, f(x)$ is the finite union of convex sets with at most $\bar{r} + 1$ of dimension greater than $\bar{r} - 1$. Then f has a fixed point.*

The results above have immediate application to the central theorem of game theory, namely, the min max theorem. Let X and Y be convex bodies in R^k and R^l and *let f be a real valued* (*continuous*) *map on* $X \times Y$. A saddle point or min max point x_0, y_0 is defined by

$$\operatorname*{Min}_{y \in Y} f(x^0, y) = f(x^0, y^0) = \operatorname*{Max}_{x \in X} f(x, y^0).$$

Define

$$M(y) = \{x | f(x, y) = \operatorname*{Max}_{x \in X} f(x, y)\} \subset X,$$

$$N(x) = \{y | f(x, y) = \operatorname*{Min}_{y \in Y} f(x, y)\} \subset Y.$$

Let $g(x, y)$ be the set valued transformation on $X \times Y$ to $X \times Y$ defined by

$$g(x, y) = M(y) \times N(x).$$

Our new type of saddle point theorem is

THEOREM 5. *Suppose M and N are usc with singular sets*

$$S(X) = \bigcup_r \mu_r(X), \qquad S(Y) = \bigcup_r \mu_r(Y).$$

Write $d(X) = \dim S(X)$, $d(Y) = \dim S(Y)$. Suppose $\mu_r(X) = \varnothing$ for $r \geqq p$ and that $dX \leqq k - p - 3$ and suppose too that $\mu_r(Y) = \varnothing$ for $r \geqq q$ and that $dY \leqq l - q - 3$. For $x \bar{\in} S(X)$, $N(X)$ is convex and for $x \in S(X)$, $N(x)$ is a finite union of convex sets at most p of which are of dimension at least $p - 1$. For $y \bar{\in} S(Y)$, $M(y)$ is convex and for $y \in S(Y)$, $M(y)$ is a finite union of convex sets at most q of which are of dimension at least $q - 1$. Then there is a saddle point.

Detailed expositions and proofs of the results above will be given in [**1**] and [**2**].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77004