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A GENERALIZED MÖBIUS INVERSION FORMULA

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1. **Introduction.** We owe to G.-C. Rota [*On the foundations of combinatorial theory* I. *Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, Band 2, (1964), 340–368] the idea of obtaining explicit formulas for the chromatic polynomials of graphs or maps by use of a so-called Möbius inversion formula. His formulas give the development of these polynomials in powers of λ, where λ is the number of colors. His work provides interesting commentary on previous work of G. D. Birkhoff [*A determinant formula for the number of ways of coloring a map*, Ann. of Math., (2) **14** (1912), 42–46] and H. Whitney [*A logical expansion in mathematics*, Bull. Amer. Math. Soc. **38** (1932), 572–579] who had discovered similar formulas by methods which, at least superficially, seemed quite different.

Now, in the case of regular planar maps, the developments in powers of $(\lambda - 2)$ or $(\lambda - 3)$ appear to have great advantages over the developments in powers of λ. General formulas for developments in powers of $(\lambda - 3)$ have so far eluded us. But general formulas for the coefficients, $a_1, a_2, \ldots$, in expansions for chromatic polynomials in the form

$$\lambda(\lambda - 1) \sum_{k=1}^{n-2} a_k(\lambda - 2)^k$$

are well known. They are given in a paper by G. D. Birkhoff and D. C. Lewis [*Chromatic polynomials*, Trans. Amer. Math. Soc., **60** (1946), 355–451], hereafter referred to as BL. The question arose whether these formulas could be obtained by use of a Möbius function for a suitably chosen partially ordered set. It seems impossible to do so by just using the Möbius function in the form given by Rota.

The purpose of this paper is to present an extremely simple generalization of the Rota-Möbius inversion formula which suffices to prove the so-called determinant formula for the chromatic polynomial for a regular planar map as given in BL, pp. 401–405, in powers of $(\lambda - 2)$. In §4, we actually give the new proof of the determinant formula, omitting the details about the so-called markings of the maps, which are supplied by appropriate passages in BL.

2. **The principal theorem.** Let S be a locally finite partially ordered set with elements $x, y, z, \ldots$. Let S_x be a finite subset of S, defined for each

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$x \in S$, such that $x \in S_x$ and such that $(y \in S_x) \supset y \leqq x$. Let $\xi(x, y) = 1$ if $x \in S_y$ and $\xi(x, y) = 0$ if x is not an element of S_y. In particular $\xi(x, x) = 1$ for all $x \in S$. Hence, by a known theorem (cf. Marshall Hall, Jr., *Combinatorial theory*, p. 15, Lemma 2.2.1), the function ξ has an inverse ψ in the so-called incidence algebra of the partially ordered set S. This means that there exists a function $\psi(x, y)$ which is 0 for all pairs (x, y) such that $x \nleqq y$ and is such that

$$\sum_{x \leqq z \leqq y} \xi(x, z)\psi(z, y) = \delta(x, y) \tag{1}$$

where $\delta(x, x) = 1$ for all x and $\delta(x, y) = 0$ if $x \neq y$. Since $\xi(x, z) = 0$ except when $x \in S_z$, we may write (1) in the form

$$\sum_{x \in S_z; z \leqq y} \xi(x, z)\psi(z, y) = \delta(x, y). \tag{2}$$

Designating the fixed elements by z and x (instead of by x and y) and using y for the index of summation, we may write the formula (2) as follows:

$$\sum_{z \in S_y; y \leqq x} \xi(z, y)\psi(y, x) = \delta(z, x). \tag{3}$$

THEOREM 1. *If g and f are any two functions defined on S such that $g(x) = \sum_{y \in S_x} f(y)$, then $f(x) = \sum_{y \leqq x} g(y)\psi(y, x)$.*

PROOF. Consider $S(x) = \sum_{y \leqq x} g(y)\psi(y, x)$. Then, using the formula giving g in terms of f and the fact that $\xi(z, y) = 1$ if $z \in S_y$, we have

$$S(x) = \sum_{y \leqq x} \left[\sum_{z \in S_y} f(z) \right] \psi(y, x) = \sum_{(y,z)}\sum f(z)\xi(z, y)\psi(y, x)$$

where the double summation sign refers to a summation over all pairs (y, z) such that $y \leqq x$ and $z \in S_y$. This sum may be displayed as an iterated sum as follows:

$$S(x) = \sum_{z \leqq x} f(z) \left[\sum_{z \in S_y; y \leqq x} \xi(z, y)\psi(y, x) \right] \cdot$$

Hence from (3) we see that

$$S(x) = \sum_{z \leqq x} f(z)\delta(z, x) = f(x).$$

The converse of this theorem is also true, and the proof (which we omit) is similarly extremely simple. Instead of being based on the formula (1) which expressed the fact that ψ is a *right* inverse of ξ, it is based on the analogous formula expressing the fact that ψ is a *left* inverse of ξ.

The Rota-Möbius inversion theorem is the special case which arises when $S_x = \{y | y \leqq x\}$, at least, assuming that S is such that S_x is finite for all x. In this case the function ξ is usually written ζ and ψ is written μ. These are the famous zeta and Möbius functions.

3. **Determination of the generalized Möbius function in a special case.** Suppose now that S is such that for each element x of S there is not more than one element y of S such that $x < y$ with no element between x and y. This means, more generally, that if $x < y$, there exists a unique maximal set of elements $z_1, z_2, \ldots, z_{k-1}$ (which may be vacuous if $k = 1$) such that $x < z_1 < z_2 < \ldots < z_{k-1} < y$. In such a case, x is said to be k steps below y. If $x = y$, x is said to be zero steps below y. We define S_y to consist of y itself together with the set of all elements x which are one step below y.

THEOREM 2. *If x is k steps below y, then $\psi(x, y) = (-1)^k$.*

PROOF. Using the fact that $\xi(x, z) = 1$ if $x \in S_z$, we write (2) in the form

$$\sum_{x \in S_z; z \leqq y} \psi(z, y) = \delta(x, y). \tag{4}$$

If $x = y$, $\delta(x, y) = 1$ and the only z, such that $z \leqq y$ and such that $x \in S_z$, is clearly $z = y$. It follows from (4) that $\psi(x, y) = 1$ if x is zero steps below y. Thus the theorem is established if $k = 0$. This result is, of course, true in the general case and not just in the special case under present consideration, if we express it as $\psi(x, x) = 1$.

Next, if x is just one step below y there are only two values of z satisfying the two conditions $z \leqq y$ and $x \in S_z$, namely $z = x$ and $z = y$. Thus the formula (4) reduces to $\psi(x, y) + \psi(y, y) = \delta(x, y) = 0$; and, since it has already been proved that $\psi(y, y) = +1$, we have $\psi(x, y) = -1$. This establishes the theorem for $k = 1$.

We now make the inductive hypothesis that the theorem is true when $k = j - 1$ and prove that it is true for $k = j$ ($j = 2, 3, \ldots$). Thus, if x is j steps below y, there exist unique elements $u_1, u_2, \ldots, u_{j-1}$ such that $x < u_1 < u_2 < \ldots < u_{j-1} < y$. Evidently $x \in S_{u_1}$, but x is not an element of S_{u_i} for $i = 2, \ldots, j - 1$; nor is x an element of S_y. These facts follow from the definition of the S's. Thus there are only two values of z satisfying the two conditions $z \leqq y$ and $x \in S_z$, namely $z = x$ and $z = u_1$, and the formula (4) reduces to $\psi(x, y) + \psi(u_1, y) = \delta(x, y) = 0$. But u_1 is $j - 1$ steps below y. Hence by our inductive hypothesis $\psi(u_1, y) = (-1)^{j-1}$. Hence $\psi(x, y) = (-1)^j$ and the induction is complete.

4. **Application to the theory of chromatic polynomials.** We apply the preceding results to obtain (without actual use of determinants) the so-called "determinant formula" for a chromatic polynomial in powers of $\lambda - 2$ (cf., BL, pp. 401-seq.). The set S is a set of maps *marked* as described in BL. The map x is one step below the map y, if x is a submap of y obtained by the erasure of an arbitrary set of *marked* boundaries of y. We

confine attention to the case where all the maps of S are obtained from a single map in a (finite) number of such steps. The maps are identified and marked as obtained, and we do not assume necessarily that two distinct maps x and y $(x \neq y)$ are necessarily geometrically distinct. The set S is always finite, so that it will contain minimal elements. These are the maps for which the set of marked boundaries must or may be taken as vacuous.

We let $P_x(\lambda)$ denote the chromatic polynomial of the marked map x. This is the number of ways the map may be colored in λ colors in the usual sense. But the number of ways the map may be colored in λ colors without regard to color clashes at the *marked* boundaries is $\lambda(\lambda - 1)(\lambda - 2)^{n_x - 2}$, where n_x is the number of regions in the marked map x. This follows from the way the boundaries are marked. We evidently have

$$\lambda(\lambda - 1)(\lambda - 2)^{n_x - 2} = \sum_{y \in S_x} P_y(\lambda)$$

where S_x is the set of all maps obtained from x in just one step plus the map x itself. So taking $f(x) = P_x(\lambda)$ and $g(x) = \lambda(\lambda - 1)(\lambda - 2)^{n_x - 2}$, we find from Theorem 1 that

$$P_x(\lambda) = \sum_{y \leqq x} \lambda(\lambda - 1)(\lambda - 2)^{n_y - 2}\psi(y, x).$$

But by Theorem 2, $\psi(y, x) = (-1)^k$, where $k = k(x, y)$ is the number of steps by which y is below x. Thus

$$\begin{aligned} P_x(\lambda) &= \lambda(\lambda - 1)\left[\sum_{y \leqq x} (-1)^{k(x,y)}(\lambda - 2)^{n_y - 2}\right] \\ &= \lambda(\lambda - 1)\left[\sum_{j=2}^{n_x}\left(\sum_{n_y = j; y \leqq x} (-1)^{k(x,y)}\right)(\lambda - 2)^{j-2}\right]. \end{aligned}$$

Let $[p \cdot j]_x$ denote the number of elements y satisfying the following conditions: $y \leqq x$, $n_y = j$, and $k(x, y) = p$. Then

$$\sum_{n_y = j; y \leqq x} (-1)^{k(x,y)} = \sum_p (-1)^p[p \cdot j]_x.$$

Substituting in the preceding formula for $P_x(\lambda)$ we obtain

$$P_x(\lambda) = \lambda(\lambda - 1)\left[\sum_{j,p} (-1)^p[p \cdot j]_x(\lambda - 2)^{j-2}\right], \tag{5}$$

where $[p \cdot j]_x$ evidently denotes the number of maps in the set S having just j regions that are obtained from the map x in just p steps. The formula (5) is essentially the formula (7.2) of BL, p. 403. The presence of the subscript x is connected with the fact that (5) holds for any map of the set S and not just for the map corresponding to the maximal element in S.

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