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WHICH ABELIAN GROUPS CAN BE FUNDAMENTAL GROUPS OF REGIONS IN EUCLIDEAN SPACES?[1]

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Let C_n denote the collection of all abelian groups that can be fundamental groups of regions in S^n. It is clear that $C_k \subseteq C_{k+1}$. It is also easy to see that C_1 and C_2 each consist of just two groups—the trivial groups 1 and the infinite cyclic group Z. We shall see in this paper that actually $C_k = C_{k+1}$ for $k \geqq 4$, so we shall be concerned mainly with the difference between regions in S^3 and regions in S^4.

If a region A in S^n is not S^n itself, we may assume that $A \subset R^n$, and that there is a point e of A that is at a distance $\geqq 1$ from $R^n - A$. Using barycentric subdivision T_k of R^n of mesh converging to zero, where T_l is a refinement of T_k if $l < k$, let U_k be the interior of the union of those simplexes that lie in A and are at a distance $\leqq k$ from e. Take A_k to be the component of U_k that contains e_i. It is easy to see that $A_l \leqq A_k$ if $l < k$, and that $\bigcup_{k=1}^{\infty} A_k = A$; thus $\pi(A)$ is equal to the direct limit of the sequence $\{\pi(A_k)\}$. Since each $\pi(A_k)$ is finitely generated, $\pi(A)$ must be countable.

Now suppose that $G = \pi(A)$ is abelian. Since $G_i = \pi(A_i)$ is finitely generated, the image K_i of G_i in some $G_s = \pi(A_s)$ of the inclusion $G_i \rightarrow G_s$ must be abelian. Replacing the sequence $\{G_i\}$ by a subsequence if necessary, we may assume that the image K_i of G_i in G_{i+1} is abelian.

The calculation of C_3 is closely related to the following problem: "Which elements of a link group commute?" In fact, if we use brick subdivision instead of barycentric subdivision of R^3 in the construction of A_k, we may assume that each $S^3 - A_k$ is the union of a finite number of handle-bodies-with-knotted-holes, semilinearly imbedded in S^3. Since each G_k is finitely generated, so is its abelianized group $\bar{G}_k = H_1(A_k)$. We can find non-singular loops $\{x_1, \ldots, x_p\}$ that generate $H_1(A_k)$. By the Alexander duality theorem and the fact that $S^3 - A_k$ is a manifold, we can also find non-singular loops $\{y_1, \ldots, y_p\}$ in $S^3 - A_k$ which are dual to $\{x_1, \ldots, x_p\}$ in the sense that the linking number (x_i, y_i) between x_i and y_i is equal to δ_{ij}, where δ_{ij} is the Kronecker delta. The image of any two elements of G_{k-1} in G_k must commute in the complement of the link $y_1 \cup y_2 \cup \cdots \cup y_p$.

The following theorem (cf. [**6**] and [**7**]) makes it possible to deal with arbitrary links.

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[1] This paper represents a portion of the author's Ph.D. thesis, written under the direction of Professor Ralph H. Fox at Princeton University.

THEOREM A. *Let l be a link with two components whose linking number is λ. Then G, the group of l,* mod G_2, *its second lower central subgroup, has the following presentation*: $\{a, b : [a, b^\lambda] = [a^\lambda, b] = 1, F_2 = 1\}$, *where F is the free group $\{a, b\}$.*

Now with the help of this theorem and free calculus (cf. [**2**]) we can prove the following theorem:

THEOREM 1. *Let $l_1, l_2, \ldots, l_n$ be components of a link l and let $a_1, a_2, \ldots, a_n$ be their meridians. If two elements x and y in $\pi(S^3 - l)$ commute, and if*[2] $x \sim a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}$, $y \sim a_1^{\beta_1} a_2^{\beta_2} \ldots a_n^{\beta_n}$, *then the linking number λ_{ij} of l_i and l_j must divide*

$$\begin{vmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{vmatrix}.$$

To calculate C_3 we also need the following theorem:

THEOREM 2. *If G is the fundamental group of a region A in S^3, then no abelian subgroup of G has rank greater than* 2.

The proof of Theorem 2 is merely a modification of an argument due to Conner (cf. [**1**, Theorem 2]). By Theorem 2 we know that each K_m must be either 1, Z, or $Z + Z$. We can now state our theorem about regions in S^3:

THEOREM 3. *An abelian group G is the fundamental group of a region A in S^3 if and only if G is* 1, *Z, $Z + Z$, or a subgroup of the additive group of rational numbers.*

It is not difficult to prove the "if" part of Theorem 3. The trivial knot has Z as its group, and the trivial link whose linking number is 1 (see Figure 1) has $Z + Z$ as its fundamental group, and the well-known P-adic

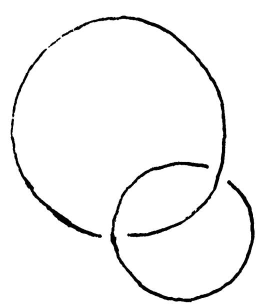

FIGURE 1

solenoid (cf. [**3**]) gives us the P-adic group as the fundamental group of its complement. The complements of other kinds of solenoids give us regions

[2] We write $f \sim g$ to denote that f is homologous to g.

whose fundamental groups are just the various subgroups of the additive group of rational numbers (cf. [**5**, Chapter VIII]). For the "only if" part, we need Theorem 1 and Theorem 2, and some other constructions and lemmas. It is too long to give here.

Before we go on to find C_4, it is interesting to see some results about a union of knotted 2-spheres in S^4. We have the following theorem, analogous to Theorem A.

THEOREM 4. *For a disjoint union of an arbitrary number of spheres in S^4, the group $G = \pi(S^4 - S_1 \cup \cdots \cup S_n)$ depends* mod G_2 *only on the number of spheres. In fact $G/G_2 = F/F_2$, where F is the free group with n generators.*[3]

Therefore except for 1 and Z, no abelian group can be the group G. Furthermore, the center C of G must be contained in the commutator subgroup G_1 of G.

The proof of Theorem 4 makes use of the method of hyperplane cross-section (cf. [**4**]). We can represent the imbedding of $S_1 \cup S_2 \cup \cdots \cup S_n$ by a family of links $l_1^t \cup l_2^t \cup \cdots \cup l_n^t$, where l_i^t is the cross section of S_i with the hyperplane $x_4 = t$, where (x_1, x_2, x_3, x_4) is the coordinate of a point in R^4. The proof needs the fact that the presentation given in Theorem A is almost canonical, and also the following theorem:

THEOREM 5. *The linking number (K_i^t, l_j^t) between a component K_i^t of l_i^t with the totality of l_j^t is always zero.*[4]

Theorem 5 has its own interest, in the sense that it gives us a necessary condition for a link to be a link sliced from n 2-spheres. In general in order that l be a slice of a union of n 2-spheres, we must be able to orient l and divide l into n links $l_1 \cup l_2 \cup \cdots \cup l_n$ in such a way that the linking number of any component of l_i with l_j for any $i \neq j$ is always zero.

Now we may state our theorem about regions in S^4.

THEOREM 6. *An abelian group G is the fundamental group of a region in S^n for $n \geqq 4$ if and only if it is countable.*

By what we said at the beginning of this paper, it is clear that we need only to prove the "if" part for the case $n = 4$, and it follows that $C_4 = C_5 = \cdots$. The only way to prove this theorem is to give an algorithm to construct a region A in S^4 (actually in R^4) whose fundamental group is a given countable abelian group G. It is too long to give here.

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[3] We assume our imbedding is semilinear and locally flat.

[4] We assume that S_i are orientated, thus orientation is induced on each l_i^t.

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