BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, May 1972

COTERMINAL FAMILIES AND THE STRONG MARKOV PROPERTY

BY A. O. PITTENGER AND C. T. SHIH

Communicated by Harry Kesten, October 27, 1971

1. Let Ω be the set of right continuous paths mapping $[0, \infty)$ into (E_Δ, ρ), where E is a locally compact, separable metric space with metric ρ and $E_\Delta = E \cup \{\Delta\}$, $\{\Delta\}$ the point at infinity. Following the notation of [**1**], we let $X = (\Omega, X_t, \mathscr{F}, \mathscr{F}_t, \theta_t, P^x)$ be a strong Markov process with stationary transition probabilities on (E_Δ, ρ), and we use $\mathscr{B}$ and $\mathscr{B}_\Delta$ to denote the σ-fields of Borel sets on E and E_Δ respectively. The σ-fields $\{\mathscr{F}_t, 0 \leqq t\}$ are assumed to be right continuous. Finally we assume a fixed initial distribution μ, and all a.s. statements will refer to $P = \int_{E_\Delta} \mu(dx) P^x$.

The purpose of this note is to outline a type of strong Markov property for a class of random times, resembling last exit times from sets. To motivate the problem, write the strong Markov property at a stopping time T as

$$P(X_t \in A \mid \mathscr{F}_T) = P^{X_T}(X_{t-T} \in A) \quad \text{a.s.} \tag{1}$$

on $\{T < t\}$, where $\mathscr{F}_T$ is the usual σ-field of information up through T. We are interested in finding an analogue of (1) with T replaced by random times such as L^t, the last exit before $(t + 0)$ from a given set, and $\mathscr{F}_T$ replaced by an appropriate σ-field. Since the conditioning now involves the future of the process, the distributions on the right of (1) will be altered but will be seen to depend only on X_{L^t} or X_{L^t-} and $t - L^t$.

As an example suppose X is reflecting Brownian motion on $[0, \infty)$, $P = P^0$ and L^t is the last hit of $\{0\}$ prior to t. We leave it to the reader to verify that if $\sigma(L^t)$ is the σ-field generated by L^t, then

$$P^0(X_t \in A \mid \sigma(L^t)) = \int_A \frac{y \exp(-y^2/2(t - L^t))\, dy}{t - L^t} \quad \text{a.s.} \tag{2}$$

on $\{0 < L^t < t\}$. Moreover another computation shows that

$$\begin{aligned} &\int_0^\infty \frac{1}{\sqrt{s}} \left(\frac{y \exp(-y^2/2s)\, dy}{s} \right) P^y(T_{\{0\}} > t, X_t \in A) \\ &\qquad = \int_A \frac{1}{\sqrt{s+t}} \left(\frac{x \exp(-x^2/2(t+s))\, dx}{s+t} \right), \end{aligned} \tag{3}$$

AMS 1970 *subject classifications*. Primary 60J25, 60J40.

i.e., the new distributions can be normalized so that they form an entrance law relative to the family of transition probabilities

$$H_t(x, A) = P^x\{X_t \in A, T_{\{0\}} > t\}.$$

The results we summarize here in essence generalize (2) and (3) to a wide class of strong Markov processes, including standard processes, and to arbitrary coterminal families $\{L^t, t \geqq 0\}$ as defined below. Two different σ-fields will be used in the analogues of (2), and the resulting families of probability distributions will depend only on X_{L^t} (or X_{L^t-}) and $t - L^t$. In both cases a suitable normalization renders the families entrance laws relative to an appropriate transition probability. Details and proofs of these results will appear elsewhere.

2. Following [3] we characterize our random times axiomatically. Let k_t be the killing operator defined by $(k_t\omega)(s) = \omega(s)$, $s < t$, and Δ elsewhere.

DEFINITION 2.1. A family $\{L^t, t \geqq 0\}$ of $\mathscr{F}$-measurable functions will be called a coterminal family if it satisfies (i) $0 \leqq L^t \circ k_t \leqq L^t \leqq t$, (ii) $L^{t-s} \circ \theta_s = (L^t - s)^+$ for $s < t$, where $a^+ = a \vee 0$, (iii) $\lim_{u \downarrow s} L^t \circ k_u = L^s$ for $s < t$, and (iv) $L^t < s < t$ implies $L^t = L^s$.

DEFINITION 2.2. L is an *exact coterminal time* if it is $\mathscr{F}$-measurable and satisfies (i) $0 \leqq L \circ k_t \leqq t$, (ii) $L \circ \theta_s = (L - s)^+$, (iii) $L \circ k_s = L$ on $\{L < s\}$ and (iv) $L = \lim_{s \to \infty} L \circ k_s$.

To avoid technical difficulties we have avoided a.s. statements and have assumed both definitions hold on all of Ω.

Coterminal times are studied in [3], and our times L^t can be defined from an exact coterminal time by $L^t = \lim_{u \downarrow t} L \circ k^u$. Conversely, $L \equiv \lim_{t \uparrow \infty} L^t$ is an exact coterminal time. Moreover, as in [3], we have

LEMMA 2.1. *There exists a perfect, exact terminal time T associated with $\{L^t, t \geqq 0\}$ such that*

$$T = \inf\{s : L^s > 0\}. \tag{4}$$

Alternatively it is possible to begin with such a terminal time and define the family $\{L^t, t \geqq 0\}$ by

$$L^t = \lim_{v \downarrow t} \sup\{u : T \circ \theta_u \circ k_v < \infty\}.$$

The duality between T and $\{L^t, t \geqq 0\}$ is canonical in the sense that T can then be retrieved via equation (4).

3. The σ-fields we use are given in

DEFINITION 3.1. The *past* of a random variable R is the σ-field $\mathscr{F}(R-)$ generated by sets $F_s \cap \{s < R\}$, $0 \leqq s < \infty$. The *past plus present* of R is the σ-field $\mathscr{F}(R)$ generated by $\mathscr{F}(R-)$ and $\sigma(X_R)$.

These definitions together with an extensive discussion are given in [**2**].

4. Let $\{L^t, t \geqq 0\}$ be a fixed coterminal family and T the associated terminal time.

DEFINITION 4.1. Let $A^r = \{x : P^x(T = 0) = 1\}$ and $A^i = \{x : P^x(T = 0) = 0\}$.

One conditional distribution is given in

DEFINITION 4.2. Let $f \in b\hat{\mathscr{E}}_\Delta$, the bounded universally measurable functions on E_Δ. If $x \in A^i$ and $s > 0$,

$$D(x, s, f) = E^x[f(X_s) | T > s],$$

with the convention throughout that $0/0 = 0$. For $x \in A^r$ let

$$D_n(x, u, s, f) = E^x[f(X_s) | \rho(x, X_u) < 1/n, T \circ \theta_u > s - u].$$

Then

$$\bar{D}(x, s, f) = \varlimsup_{n \to \infty} \varlimsup_{u \to 0} D_n(x, u, s, f),$$

and

$$\underline{D}(x, s, f) = \varliminf_{n} \varliminf_{u} D_n(x, u, s, f),$$

where $u \to 0$ through the rationals. If $\bar{D}(x, s, f) = \underline{D}(x, s, f)$, we say $D(x, s, f)$ exists and denotes the common value.

Then

THEOREM 1. *If $f \in b\hat{\mathscr{E}}_\Delta$, almost surely on $\{0 \leqq L^t < t\}$, $D(X_{L^t}, t - L^t, f)$ exists and*

$$E[f(X_t) | \mathscr{F}(L^t)] = D(X_{L^t}, t - L^t, f).$$

5. Conditioning by $\mathscr{F}(L^t -)$ requires a different distribution. Let $f, h \in b\hat{\mathscr{E}}_\Delta$. Then

DEFINITION 5.1. If $x \in A^i$, $Q^h(x, s, f) = E^x[h(x) \cdot f(X_s) | T > s]$. If $x \in A^r$, let $Q^h(x, u, s, f) = E^x[h(X_u) \cdot f(X_s) | T \cdot \theta_u > s - u]$ and let $\bar{Q}^h(x, s, f)$ ($\underline{Q}^h(x, s, f)$) be the lim sup (lim inf) as $u \to 0$ through the rationals. If the two are equal, we say $Q^h(x, s, f)$ exists and denotes the common value.

To prove Theorem 2 some control over left-hand limits is essential, and to simplify the presentation we assume henceforth that X is a standard process. We emphasize, however, that Theorem 1 and a version of Theorem 2 are valid when branching points are permitted.

THEOREM 2. *If $f \in b\hat{\mathscr{E}}_\Delta$ and h is bounded continuous, $Q^h(X_{L^t -}, t - L^t, f)$ exists almost surely on $\{0 < L^t < t\}$ and*

$$E[h(X_{L^t}) \cdot f(X_t) | \mathscr{F}(L^t -)] = Q^h(X_{L^t -}, t - L^t, f).$$

As is to be expected Q^h and D are related. In fact for both f and $h \in b\hat{\mathscr{E}}_\Delta$ we have

THEOREM 3. *There exists a family of probability measures* $\{\eta(x, s, dz), (x, s) \in E_\Delta \times (0, \infty)\}$ *such that, for* $B \in \mathscr{B}_\Delta$, $\eta(\cdot, \cdot, B)$ *is universally measurable in the product space* $(E_\Delta \times (0, \infty))$ *and such that*

$$E[h(X_{L^t}) \cdot f(X_t) | \mathscr{F}(L^t -)] = \int_{E_\Delta} \eta(X_{L^t -}, t - L^t, dz) D(z, t - L^t, f) h(z)$$

a.s. on $\{0 < L^t < t\}$.

6. If $\mathscr{H}$ is a countable collection of functions in $b\hat{\mathscr{E}}_\Delta$, then by Theorem 1 almost surely on $\{0 \leqq L^t < t\}$

$$(X_{L^t}, t - L^t) \in \bigcap_{f \in \mathscr{H}} \{(x, s) : D(x, s, f) \text{ exists}\}.$$

Choosing an appropriate $\mathscr{H}$ including a dense subset of the continuous functions on E_Δ, we obtain for each (x, s) in the intersection above a Borel measure $D(x, s, \cdot)$ such that

$$D(x, s, f) = \int D(x, s, dy) f(y)$$

for all $f \in \mathscr{H}$. A set $G_D \subset E_\Delta$ is then defined such that for each $x \in G_D$ the measures $D(x, s, \cdot)$ can be extended to all $s > 0$ and normalizing parameters $d(x, s)$ defined so that $\{D(x, s, \cdot)/d(x, s), 0 < s < \infty\}$ form an entrance law relative to the family of transition probabilities $H_t(x, f) = E^x[T > t; f(X_t)]$. In addition $X_{L^t} \in G_D$ a.s. P^μ for each μ and $t > 0$ on $\{0 \leqq L^t < t\}$.

An identical analysis holds for Q^h, and we have completed the analogy with (3) above. Moreover we can go one step further and use the entrance law property to prove the following version of the strong Markov property for L^t.

THEOREM 4. *Let* $f_i \subset b\hat{\mathscr{E}}_\Delta$, $1 \leqq i \leqq n + 1$, *and* $0 < s_1 < \cdots < s_n < t$. *If* $\bar{t}(\omega) \equiv t - L^t(\omega)$, *then almost surely, on* $\{0 \leqq L^t < t - s_n\}$,

$$E\left[f_{n+1}(X_t) \prod_{k=1}^{n} f_k(X_{L^t + s_k}) | \mathscr{F}(L^t) \right]$$
$$= \int \cdots \int \frac{D(X_{L^t}, s_1, dz)}{D(X_{L^t}, s_1, H_{\bar{t} - s_1})} f_1(z_1) H_{s_2 - s_1}(z_1, dz_2) \cdots$$
$$f_n(z_n) H_{\bar{t} - s_n}(z_n, f_{n+1}).$$

If $\mathscr{F}(L^t)$ *is replaced by* $\mathscr{F}(L^t -)$, *a similar expression holds with D replaced by* Q^1 *and* $X_{L^t -}$ *by* X_{L^t}.

7. In the event $\{L^t, t \geqq 0\}$ is generated by a nontrivial exact coterminal time L, all of the foregoing results can be extended to L. Two families of probability distributions Q_L^h and D_L can be shown to exist and to satisfy the analogues of Theorems 1, 2 and 3. As in §6 both families define entrance laws, here relative to

$$K_t(x, f) = E^x[f(X_t)|T = \infty].$$

Furthermore that property can be used to establish a strong Markov property of the form given in Theorem 4 with $t = \infty$, and H replaced by K.

References

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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104