BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 3, May 1971

A REMARK ON CLASSIFICATION OF RIEMANNIAN MANIFOLDS WITH RESPECT TO $\Delta u = Pu$

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Communicated by W. H. Fuchs, October 30, 1970

Consider an orientable C^∞ Riemannian manifold R of dimension $m \geqq 2$ and an elliptic partial differential equation $\Delta u = Pu$ on R with P a nonnegative, not identically zero, C^1 function on R. We denote by O_{PX} the set of pairs (R, P) such that the subspace PX(R) of the space $P(R)$ of solutions u of $\Delta u = Pu$ on R determined by a property X reduces to $\{0\}$. Here the possibilities for X that we consider are B (boundedness), D (Dirichlet-finite: $D_R(u) = \int_R du \wedge * du < \infty$), E (energy-finite: $E_R(u) = \int_R (du \wedge * du + Pu^2 * 1) < \infty$), and their combinations BD, BE. The purpose of this note is to announce the following complete inclusion relations:

$$(1) \qquad O_G < O_{PB} < O_{PD} = O_{PBD} < O_{PE} = O_{PBE}.$$

The symbol $\mathfrak{A} < \mathfrak{B}$ means that $\mathfrak{A}$ is a proper subset of $\mathfrak{B}$ and O_G is the set of pairs (R, P) such that R does not possess a harmonic Green's function. This type of classification of Riemannian manifolds was initiated by Ozawa [**9**]. Myrberg [**3**] demonstrated the existence of the Green's function of $\Delta u = Pu$ on R and thus also the existence of a positive solution of $\Delta u = Pu$ on R without any further restrictions on P. One of the most interesting results of Ozawa is:

$$(2) \qquad O_{PB} = O_{PD} = O_{PBD} = O_{PE} = O_{PBE}$$

if the pairs (R, P) are required to satisfy $\int_R P * 1 < \infty$. This condition was weakened by Glasner-Katz [**1**] as follows: (2) is valid for (R, P) such that there exists a subregion $R' \notin SO_{HD}$ (cf. [**11**]) with $\int_{R'} P * 1 < \infty$. That this is the weakest condition under which (2) is valid was shown by Glasner-Katz-Nakai [**2**].

AMS 1970 *subject classifications*. Primary 30A48, 31B05, 35J05, 53C20.

Key words and phrases. Riemann surface, Riemannian manifold, harmonic function, solution of $\Delta u = Pu$, Dirichlet integral, energy integral, classification theory of Riemann surfaces.

[1] Supported in part by NSF Grant No. 14133, California Institute of Technology.

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[3] Supported by the U. S. Army Research Office-Durham, Grant DA-ARO-D-31-124-71-G20, UCLA.

The first inclusion, $O_G \subset O_{PB}$, in (1) was established by Ozawa [9] and also Royden [10]. Its strictness was observed by Royden [10] for the case $m=2$. The higher dimensional example is trivially furnished by (E^m, P), where E^m is m-dimensional Euclidean space and P increases rapidly at the point at ∞ (see (6) below).

The second inclusion, $O_{PB} \subset O_{PD}$, was originally observed by Nakai [5]. Recently Nakai [6], [7] showed that $O_{PD}=O_{PBD}$ from which the latter inclusion also follows trivially. The strictness of this inclusion, $O_{PB} < O_{PD}$, was established in [2] for every $m \geqq 2$ except $m=3$. This gap will be filled here and in the process the strictness of $O_{PD} \subset O_{PE}$ will also be settled.

Let $P(x)$ be a rotation invariant nonnegative C^1 function on E^3 which is equal to $1/|x|^{2+\alpha}$ $(\alpha>0)$ near ∞. We denote by $\mathrm{HX}(E^3)$ the space of harmonic functions on E^3 with the property X (X=B, D, BD, BE). Clearly $\mathrm{HX}(E^3)=E^1$. Since every PB-function is the difference of two nonnegative ones, if $\mathrm{PB}(E^3) \neq \{0\}$, then we can find $u \in \mathrm{PB}(E^3)$ with $u>0$. In view of $\mathrm{HX}(E^3)=E^1$ we have

$$(3) \quad u(x) = c - \frac{1}{4\pi}\int_{E^3} \frac{1}{|x-y|} P(y)u(y)\, dy, \qquad dy = dy^1 dy^2 dy^3,$$

where c is a constant with $c>u$. Take another such $\bar{u}$ and let $\bar{c}$ be the corresponding constant. Then

$$|w| \leqq \frac{1}{4\pi}\int_{E^3} \frac{1}{|x-y|} P(y)\,|w(y)|\, dy,$$

where $w=\bar{c}u-c\bar{u}$. Since $|w|$ is subharmonic and the right-hand side is a potential, we must have $|w|=0$. Therefore $\mathrm{PB}(E^3)$ is one-dimensional. This together with the fact that $\Delta u=Pu$ is invariant under rotations gives the fact that every $u \in \mathrm{PB}(E^3)$ is also invariant under rotations. Thus if $u>0$, then by the maximum principle

$$(4) \qquad \inf u > 0.$$

We conclude that $(E^3, P) \notin O_{PB}$ if and only if

$$(\mathrm{B}) \qquad \int_{E^3} \frac{1}{|x-y|} P(y)\, dy < \infty.$$

Observe that

$$D_{E^3}(u) = \frac{1}{4\pi}\int_{E^3\times E^3} \frac{1}{|x-y|} P(x)P(y)u(x)u(y)\, dx\, dy,$$

and hence from (4) it can be seen that $(E^3, P)\notin O_{PBD}=O_{PD}$ if and only if

$$\text{(D)}\qquad \int_{E^3\times E^3} \frac{1}{|x-y|} P(x)P(y)\,dx\,dy < \infty.$$

From (2), (4) and (B) it is also clear that $(E^3, P)\notin O_{PBE}=O_{PE}$ if and only if

$$\text{(E)}\qquad \int_{E^3} P(x)\,dx < \infty.$$

Since $\alpha>0$, (B) is valid, and a fortiori $(E^3, P)\notin O_{PB}$. By virtue of

$$\text{(5)}\qquad \int_{E^3} \frac{1}{|x-y|}\,\frac{1}{|y|^{2+\alpha}}\,dy \sim \frac{1}{|x|^{\alpha}} \qquad (|x|\to\infty), \qquad 0<\alpha<1,$$

(D) is false if, for example, $\alpha=0.5$. Here $a(x)\sim b(x)$ $(|x|\to\infty)$ means that $k^{-1}b(x)\leqq a(x)\leqq kb(x)$ for every x with $|x|$ sufficiently large and a constant $k\geqq 1$. Thus we have $(E^3, P)\in O_{PD}-O_{PB}$ when $\alpha=0.5$.

The equality $O_{PE}=O_{PBE}$ was established by Royden [**11**]. The inclusion $O_{PD}=O_{PBD}\subset O_{PBE}=O_{PE}$ is seen to be strict if α is chosen so as to make (E) invalid but (D) valid. For example $(E^3, P)\in O_{PE}-O_{PD}$ if $\alpha=0.9$.

Let $G(x, y)$ be the harmonic Green's function on the Riemannian manifold R. Then condition (B) takes the form $\int_R G(x, y)P(y)\,dy$ and is known to be sufficient for PB(R) and HB(R) to be isomorphic as vector spaces (cf. [**4**], [**6**], [**7**]). Similarly, the analogue of condition (D), $\int_{R\times R} G(x, y)P(x)P(y)\,dx\,dy<\infty$, is sufficient for the isomorphism of PBD(R) and HBD(R) (cf. [**6**], [**7**]). Royden [**10**] showed that $\int_R P * 1<\infty$, i.e. condition (E) gives the isomorphism of PBE(R) and HBD(R). The converses of these statements are not true in general. For example, if $R=\{x\in E^2 \mid 0<|x|<1\}$ and $P(x)=|x|^{-2}$, then PX(R) is isomorphic with HX(R) (X=B, BD, BE) yet each of (B), (D) and (E) is invalid. However, if we avoid a neighborhood of $x=0$ a nonessential boundary point of R in the integrations by considering $R'=\{x\in E^2 \mid 0<\epsilon<|x|<1\}$, we regain the validity of (B), (D) and (E). Thus in some sense the converses seem to be "almost" true. Since the key to the counterexamples given here is the validity of the converses of these "comparison theorems" in the above setting, we feel that their formulation in the general situation deserves to be pursued.

The above examples of course generalize to E^m with $m\geqq 3$ arbitrary. Actually more is known (cf. Nakai [**8**]): Let $P_\alpha(x)$ be a nonnegative

C^1 function on E^m ($m \geqq 3$) such that $P_\alpha(x) \sim |x|^{-\alpha}$ ($|x| \to \infty$). Then

$$(6)\qquad \begin{array}{ll} (E^m, P_\alpha) \in O_{PB} - O_G & (\alpha \leqq 2); \\ (E^m, P_\alpha) \in O_{PD} - O_{PB} & (2 < \alpha \leqq (m+2)/2); \\ (E^m, P_\alpha) \in O_{PE} - O_{PD} & ((m+2)/2 < \alpha \leqq m); \\ (E^m, P_\alpha) \notin O_{PE} & (m < \alpha). \end{array}$$

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ADDED IN PROOF. Recently Nakai [*A remark on classification of Riemann surfaces with respect to $\Delta u = Pu$*, Bull. Amer. Math. Soc. **77** (1971), (to appear).] showed $O_{PD} < O_{PE}$ for Riemann surfaces, and also he [*The equation $\Delta u = Pu$ on the unit disk with almost rotation free $P \geqq 0$* (to appear)] proved an analogue of (6) for $\{|z| < 1\}$ and $P_\alpha(z) \sim (1 - |z|)^{-\alpha}$ ($|z| \to 1$). Therefore (1) is true for every dimension $m \geqq 2$.