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A NOTE ON COBORDISM OF POINCARÉ DUALITY SPACES

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1. **Introduction.** Let Ω_n^{PD} denote the group of cobordism classes of oriented Poincaré duality spaces of dimension n. (See [2] for definitions.) The Pontrjagin-Thom construction yields a natural homomorphism $p:\Omega_n^{\mathrm{PD}}\to\pi_n(MSG)$ where MSG is the Thom spectrum associated to the universal spherical fibration over BSG.

N. Levitt [2] has shown that if $n\not\equiv 3 \pmod 4$, then p is surjective, and if $n\equiv 3 \pmod 4$, then cokernel$(p)\subseteq \mathbf{Z}_2$. More precisely, Levitt has shown that, if $n\geqq 3$, there is a subgroup $\bar{\Omega}_n\subseteq\Omega_n^{\mathrm{PD}}$ (it is likely that $\bar{\Omega}_n=\Omega_n^{\mathrm{PD}}$) and an exact sequence

$$(1.0)\qquad \cdots \to P_n \to \bar{\Omega}_n \xrightarrow{p} \pi_n(MSG) \to P_{n-1} \to \cdots$$

where $P_n=\mathbf{Z}$, 0, $\mathbf{Z}_2$, 0 as $n\equiv 0$, 1, 2, 3 (mod 4), respectively. Further, image$(P_n)\subset\Omega_n^{\mathrm{PD}}$ is generated by the cobordism class $[K^n]$ where, if $n\equiv 0 \pmod 4$, K^n is the almost parallelizable Milnor manifold of index 8, and, if $n\equiv 2 \pmod 4$, K^n is the almost parallelizable Kervaire manifold constructed by plumbing together the tangent bundles of two $(n/2)$-spheres. (K^4 is not a manifold, but it is a Poincaré duality space.)

Our main results, proved in §2, are the following.

THEOREM 1.1. *The Kervaire manifold, K^{4k+2}, bounds a Poincaré duality space.*

THEOREM 1.2. *The Milnor manifold, K^{4k}, is Poincaré duality cobordant to* $8(\mathbf{CP}(2))^k$.

It follows from Theorem 1.1 that the long exact sequence (1.0) contains short exact sequences

$$0 \to \bar{\Omega}_{4k+3} \to \pi_{4k+3}(MSG) \to \mathbf{Z}_2 \to 0.$$

Our proof of Theorem 1.1 can be formulated to show that this sequence is actually split exact.

Theorem 1.2 describes the short exact sequences

$$0 \to \mathbf{Z} \to \bar{\Omega}_{4k} \to \pi_{4k}(MSG) \to 0$$

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which occur in (1.0). For, $\bar{\Omega}_{4k}$ is a direct sum of Z and the subgroup of $\bar{\Omega}_{4k}$ of elements of index zero and $[(CP(2))^k]$ can be chosen as a generator of the summand Z.

Since it is not known if $\bar{\Omega}_n = \Omega_n^{\mathrm{PD}}$, it does not follow immediately that the cokernel of $p:\Omega_{4k+3}^{\mathrm{PD}} \to \pi_{4k+3}(MSG)$ is Z_2. However, this is, in fact, the case and in §3 we outline a second proof of Theorem 1.1 (actually, the original proof), due to the first-named author of this note, which shows this additional fact.

2. **Proof of Theorems 1.1 and 1.2.** Suppose given a diagram

$$\begin{array}{ccc} \nu_N & \xrightarrow{\hat{f}} & \xi_M \\ \downarrow & & \downarrow \\ N^n & \xrightarrow{f} & M^n \end{array}$$

where N^n and M^n are closed, oriented manifolds, ν_N is the normal bundle of N^n, ξ_M is a bundle fibre homotopy equivalent to the normal bundle of M^n, f is a map of degree one, and $\hat{f}$ is a bundle map covering f. Then there is associated a surgery obstruction, $s(N^n, \hat{f}) \in P_n$, to constructing a homotopy equivalence cobordant to $(N^n, \hat{f})$. The surgery obstruction, s, satisfies the following product formula of Sullivan [3]:

$$s(L^{4k} \times N^n, 1 \times \hat{f}) = \mathrm{index}(L^{4k}) \cdot s(N^n, \hat{f}).$$

If $n \equiv 0 \pmod 4$, then $s(N^n, \hat{f}) = (1/8)\ (\mathrm{index}(N^n) - \mathrm{index}(M^n))$.

Now, Theorem 1.1 is obvious if $k=0$ since $K^2 = S^1 \times S^1 = T^2$. Also, there is a well-known normal map

$$\begin{array}{ccc} \nu_{T^2} & \xrightarrow{\hat{f}} & \nu_{S^2} \\ \downarrow & & \downarrow \\ T^2 & \xrightarrow{f} & S^2 \end{array}$$

with $s(T^2, \hat{f}) = 1$. Then $s(CP(2k) \times T^2, 1 \times \hat{f}) = 1$, and the technique of surgery can be used to construct a (normal) cobordism from $1 \times f: CP(2k) \times T^2 \to CP(2k) \times S^2$ to $g: W^{4k+2} \to CP(2k) \times S^2$ where W^{4k+2} is the connected sum of K^{4k+2} with a PL manifold V^{4k+2} homotopy equivalent to $CP(2k) \times S^2$. Clearly, W^{4k+2} is a smooth boundary since it is cobordant to $CP(2k) \times T^2$. But V^{4k+2} bounds a Poincaré duality space because it is homotopy equivalent to $CP(2k) \times S^2$. Thus, the difference $[W^{4k+2}] - [V^{4k+2}] = [K^{4k+2}] = 0$. This proves Theorem 1.1.

For Theorem 1.2, we distinguish the cases $k=1$ and $k>1$. If $k=1$, this has been shown by Wall, and follows from the fact that the index homomorphism $\Omega_4^{PD} \to \boldsymbol{Z}$ is an isomorphism. If $k>1$, we proceed as follows. Let H denote the canonical complex line bundle over $CP(2)$. Then $24H$ is fibre homotopically trivial. Hence, there is a manifold N^4 and a diagram

$$\begin{array}{ccc} \nu_N & \xrightarrow{\hat{f}} & \xi \\ \downarrow & & \downarrow \\ N^4 & \xrightarrow{\hat{f}} & CP(2) \end{array}$$

where $\xi = \nu_{CP(2)} - 24H$. By the Hirzebruch index theorem, index$(N^4) = 9$, and hence $s(N^4, \hat{f}) = (1/8)(\text{index}(N^4) - \text{index}(CP(2))) = 1$. Also, N^4 is smoothly cobordant to $9(CP(2))$. By the product formula, $s((CP(2))^{k-1} \times N^4, 1 \times f) = 1$. Again by surgery, $1 \times f: (CP(2))^{k-1} \times N^4 \to (CP(2))^{k-1} \times CP(2) = (CP(2))^k$ is cobordant to $g: W^{4k} \to (CP(2))^k$ where W^{4k} is the connected sum of K^{4k} and a PL manifold V^{4k} homotopy equivalent to $(CP(2))^k$. Since W^{4k} is smoothly cobordant to $(CP(2))^{k-1} \times N^4$, hence to $9(CP(2))^k$, and since V^{4k} is Poincaré duality cobordant to $(CP(2))^k$, it follows that the difference $[W^{4k}] - [V^{4k}] = [K^{4k}]$ is Poincaré duality cobordant to $8(CP(2))^k$. This proves Theorem 1.2.

3. **Additional comments.** Let $K(\mathbf{Z}_2, 2k+1) \to BSG\langle v_{2k+1}\rangle \to BSG$ be the fibration which kills the Wu class $v_{2(k+1)} \in H^{2(k+1)}(BSG, \mathbf{Z}_2)$ [1]. Let $MSG\langle v_{2(k+1)}\rangle$ be the Thom spectrum associated to the universal bundle pulled back to $BSG\langle v_{2(k+1)}\rangle$. If M^{4k+3} is a Poincaré duality space, then $v_{2(k+1)}(M^{4k+3}) = 0$; hence the classifying map for the normal spherical fibration, $M^{4k+3} \to BSG$, lifts to a map $M^{4k+3} \to BSG\langle v_{2(k+1)}\rangle$. It follows that if the Pontrjagin-Thom homomorphism $p: \Omega_{4k+3}^{PD} \to \pi_{4k+3}(MSG)$ is surjective, then the natural homomorphism $\pi_{4k+3}(MSG\langle v_{2(k+1)}\rangle) \to \pi_{4k+3}(MSG)$ is also surjective.

In [1] it is shown that there is an exact sequence

$$0 \to \boldsymbol{Z}_2 \xrightarrow{i} \pi_{4k+3}(MSG, MSG\langle v_{2(k+1)}\rangle) \xrightarrow{j} H_{2k+1}(MSG, \boldsymbol{Z}_2) \to 0.$$

It can further be shown that

$$\text{image}(\pi_{4k+3}(MSG) \to \pi_{4k+3}(MSG, MSG\langle v_{2(k+1)}\rangle)) = i(\boldsymbol{Z}_2) = \boldsymbol{Z}_2.$$

In particular, $\pi_{4k+3}(MSG\langle v_{2(k+1)}\rangle) \to \pi_{4k+3}(MSG)$ is not surjective; hence $p: \Omega_{4k+3}^{PD} \to \pi_{4k+3}(MSG)$ is not surjective.

This argument provides a homotopy theoretic description of Levitt's obstruction to transversality, $\pi_{4k+3}(MSG) \to \boldsymbol{Z}_2$, which occurs in the exact sequence (1.0). Namely, with the identification of their images, Levitt's homomorphism coincides with the homomorphism $\pi_{4k+3}(MSG) \to \pi_{4k+3}(MSG, MSG\langle v_{2(k+1)}\rangle)$.

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