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ABBREVIATING PROOFS BY ADDING NEW AXIOMS

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The purpose of this note is to state precisely and prove the following informal statement: If T is a theory and α is a new axiom such that $T+\text{non}\,\alpha$ is an undecidable theory then some theorems of T have much shorter proofs in $T+\alpha$ than in T. Notice that if T is an essentially undecidable theory, like e.g. arithmetic, this conclusion will be true provided α is a sentence which is not a theorem of T, since then $T+\text{non}\,\alpha$ is undecidable.

Let T be a formalized theory which among its logical functors has the negation $\neg$, the implication $\rightarrow$, and the alternative $\vee$. Let σ and τ be variables ranging over sentences formulated in the language of T and α one fixed such sentence. We denote by $\ulcorner\sigma\urcorner$ the Gödel number of σ, although here $\ulcorner\ \urcorner$ is just any one-to-one map of the set of sentences into the set of positive integers. For any theorem τ of T let $W(\tau)$ be also a positive integer measuring in some way the length of the shortest proof of τ in T. But all we need about $\ulcorner\ \urcorner$ and W are the following conditions:

(i) The set $\{2^n(2\ulcorner\tau\urcorner+1): \tau \text{ is valid in } T \text{ and } W(\tau)\leqq n\}$ is recursive.

(ii) There are recursive functions g and h such that for every σ

$$W(\alpha\rightarrow(\alpha\vee\sigma)) \leqq g(\ulcorner\sigma\urcorner), \qquad h(\ulcorner\sigma\urcorner) = \ulcorner\alpha\vee\sigma\urcorner.$$

The meaning of (i) is that there is an algorithm to check if τ has a proof of length $\leqq n$. This stipulation entails that the set of Gödel numbers of the theorems of T is recursively enumerable. It is clear that reasonable $\ulcorner\ \urcorner$ and W satisfy (i) and (ii).

LEMMA. *If the theory $T+\neg\alpha$ is undecidable, i.e. the set $\{\ulcorner\sigma\urcorner : \alpha\vee\sigma$ is valid in $T\}$ is not recursive, then there is no recursive function f such that*

$$W(\tau) \leqq f(W(\alpha\rightarrow\tau)) \tag{1}$$

for every τ valid in T.

PROOF. Suppose to the contrary that (1) holds. We can assume without loss of generality that f is nondecreasing. Then by (1) and (ii) we get

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$$W(\alpha \vee \sigma) \leqq f(W(\alpha \to (\alpha \vee \sigma))) \leqq f(g(\ulcorner\sigma\urcorner)).$$

Hence if we want to check for a given positive integer k if $k \in \{\ulcorner\sigma\urcorner : \alpha\vee\sigma$ is valid in $T\}$ it is enough to evaluate $f(g(k))$, $h(k)$ and check if

$$2^{f(g(k))}(2h(k)+1) \in \{2^n(2\ulcorner\tau\urcorner+1) : \tau \text{ is valid in } T \text{ and } W(\tau) \leqq n\}.$$

By (i) this constitutes a decision procedure, contrary to the supposition of the Lemma. Q.E.D.

To apply this Lemma to the theory $T+\alpha$ we must assume that the function $W^*(\tau)$ measuring the length of the shortest proof of τ in $T+\alpha$ is such that

(iii) There exists a recursive function r such that

$$W^*(\tau) \leqq r(W(\alpha \to \tau))$$

for every τ valid in T.

This again is true for any α and most reasonable W and W^* we can think of.

THEOREM. *If the theory $T+\neg\alpha$ is undecidable then there is no recursive function s such that*

$$W(\tau) \leqq s(W^*(\tau)) \tag{2}$$

for every theorem τ of T.

PROOF. Suppose to the contrary that (2) holds. We can assume without loss of generality that s is nondecreasing. Then by (2) and (iii) we get

$$W(\tau) \leqq s(r(W(\alpha \to \tau))),$$

which contradicts our Lemma. Q.E.D.

NOTE ADDED ON OCTOBER 25, 1970. See M. A. Arbib, *Theories of abstract automata*, Prentice-Hall, Inc. 1969, Chapter 7.4, pp. 261–267, for related results and references.

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