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A FIXED POINT THEOREM FOR PLANE CONTINUA[1]

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ABSTRACT. In this paper it is proved that every bounded arcwise connected plane continuum which does not separate the plane has the fixed point property.

A set X is said to have the fixed point property if each continuous function f on X into itself leaves some point fixed (that is, there is a point x belonging to X such that $f(x)=x$). The problem "Must a bounded plane continuum which does not separate the plane have the fixed point property?" has motivated a great deal of research in plane topology. K. Borsuk in 1932 proved that every Peano continuum which lies in the plane and does not separate the plane has the fixed point property [2]. Since that time, other general conditions have been found which insure that a plane continuum has this property. In 1967, H. Bell proved that every bounded plane continuum which does not separate the plane and has a hereditarily decomposable boundary has the fixed point property [1]. The following question is still outstanding. If a bounded plane continuum is arcwise connected and does not separate the plane, then must it have the fixed point property? Here an affirmative answer is given to this question by proving the following theorem. If M is a bounded arcwise connected plane continuum which does not have infinitely many complementary domains, then the boundary of M does not contain an indecomposable continuum.

Throughout this paper S is the set of points of a simple closed surface (that is, a 2-sphere).

The proof of the following theorem is based on techniques which are closely related to the folded complementary domain concept defined by F. Burton Jones [3, p. 173].

THEOREM 1. *Suppose M is a continuum in S, $S-M$ does not have*

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infinitely many components, and Bd M *(boundary of* M*) contains an indecomposable continuum* I. *Then every subcontinuum of* M *which contains a nonempty open subset of* I *must contain* I.

PROOF. Suppose there exist a subcontinuum F of M, a nonempty open subset G of I, and a point z of I such that z does not belong to F and G is contained in F. Let x and y be points of I such that x, y, and z lie on distinct composants of I. Let $U_1, U_2, U_3, \cdots$ and $V_1, V_2, V_3, \cdots$ be monotone descending sequences of circular regions in S centered on and converging to x and y respectively such that z belongs to $S-\mathrm{Cl}(U_1\cup V_1)$ (the complement of the closure of $U_1\cup V_1$ in S) and $\mathrm{Cl}\,U_1\cap \mathrm{Cl}\,V_1=\varnothing$. Suppose M has exactly α complementary domains (α is a natural number). There exists a sequence $K_1, K_2, K_3, \cdots$ of mutually exclusive continua such that for each positive integer n, the set $L_n=\bigcup_{i=1}^{2\alpha+2}K_{(2\alpha+2)(n-1)+i}$ is contained in $I-(U_n\cup V_n)$ and each component of L_n meets both Bd U_n and Bd V_n. Assume without loss of generality that the sequence $K_1, K_2, K_3, \cdots$, is such that for each positive integer n, there exist arc-segments $R_1^n, R_2^n, \cdots, R_\alpha^n$ and $E_1^n, E_2^n, \cdots, E_\alpha^n$ such that, for $i=1, 2, \cdots, \alpha$,

(1) $R_i^n\subset \mathrm{Bd}\ U_n$,

(2) $E_i^n\subset \mathrm{Bd}\ V_n$,

(3) R_i^n and E_i^n both meet L_n only in $K_{(2\alpha+2)(n-1)+2i}$ and

(4) R_i^n and E_i^n each have one endpoint in $K_{(2\alpha+2)(n-1)+2i-1}$ and the other endpoint in $K_{(2\alpha+2)(n-1)+2i+1}$.

The limit superior of $K_1, K_2, K_3, \cdots$ is a continuum in I which contains x and y. Since x and y do not belong to the same composant of I, the sequence $K_1, K_2, K_3. \cdots$ must converge to I.

There exists a sequence of points $z_1, z_2, z_3, \cdots$ of $I-\mathrm{Cl}\ (U_1\cup V_1)$ converging to z such that for each positive integer i, z_i belongs to K_{2i}. Let $Z_1, Z_2, Z_3, \cdots$ be a sequence of circular regions in $S-\mathrm{Cl}\ (U_1\cup V_1)$ such that

(1) for each positive integer i, Z_i is centered on z_i and has diameter less than i^{-1}, and

(2) if z_i belongs to L_n (for some integer n), then Z_i meets only the z_i-component of L_n.

Since I is in the boundary of M, for each positive integer i, there exists a point w_i which belongs to $Z_i\cap(S-M)$. For each positive integer n, there exist points c_n and d_n and arc-segments R_n and E_n such that

(1) c_n and d_n are in $\{w_{(\alpha+1)(n-1)+i} \mid i=1, 2, \cdots, \alpha+1\}$ and belong to the same complementary domain of M;

(2) $R_n \cup E_n$ belongs to the set $\{R_1^n \cup E_1^n, R_2^n \cup E_2^n, \cdots, R_\alpha^n \cup E_\alpha^n\}$; and

(3) $R_n \cup E_n \cup I$ separates c_n from d_n in S.

It follows that for each positive integer n, either $R_n \cup I$ or $E_n \cup I$ must separate c_n from d_n in S [4, Theorem 20, p. 173]. Since either $R_n \cup I$ separates c_n from d_n for infinitely many positive integers n or $E_n \cup I$ separates c_n from d_n for infinitely many n, it may be assumed without loss of generality that $R_n \cup I$ separates c_n from d_n for each positive integer n. Let $W_1, W_2, W_3, \cdots$ be a monotone descending sequence of circular regions in S centered on and converging to z such that $\mathrm{Cl}\, W_1 \cap \mathrm{Cl}\, U_1 = \varnothing$. Assume without loss of generality that for each positive integer n, the points c_n and d_n belong to W_n.

For each positive integer n, define an arc T_n and a point x_n as follows. There exists an arc B_n in $S-M$ from c_n to d_n. Let D_n denote the complementary domain of $R_n \cup I$ which contains d_n. Let k_n be the first point of $B_n \cap (\mathrm{Bd}\, W_n \cap D_n)$ and let h_n be the last point of $B_n \cap \mathrm{Bd}\, W_n$ which precedes k_n. Since $R_n \cup I$ separates h_n from k_n in S, there exists a continuum H_n in $(R_n \cup I) - W_n$ which separates h_n from k_n in $S - W_n$ [4, Theorem 27, p. 177]. Let T_n be the subarc of B_n which has endpoints h_n and k_n. Note that $T_n \cap \mathrm{Cl}\, W_n = \{h_n, k_n\}$. Let Y_1^n and Y_2^n be the mutually exclusive arc-segments in $\mathrm{Bd}\, W_n$ which have endpoints h_n and k_n. For $i=1$ and 2, there exists a point y_i^n in $H_n \cap Y_i^n$. The points y_1^n and y_2^n are contained in distinct components of $H_n - R_n$ [4, Theorem 28, p. 156]. For $i=1$ and 2, let g_i^n be a point of $\mathrm{Cl} R_n \cap \mathrm{Cl}(y_i^n\text{-component of } H_n - R_n)$. The set ($\theta$-curve) $T_n \cup \mathrm{Bd}\, W_n$ separates g_1^n from g_2^n in S. The simple closed curve $T_n \cup Y_1^n$ separates a point x_n of $\{g_1^n, g_2^n\}$ from x in S.

For each positive integer n, T_n separates X_n, the x_n-component of $I - W_n$, from A_n, the x-component of $I - W_n$ in $S - W_n$. Since x and z belong to distinct composants of I, the sequences $X_1, X_2, X_3, \cdots$ and $A_1, A_2, A_3, \cdots$ converge to I. Hence there exists a positive integer n such that $X_n \cap G \neq \varnothing$, $A_n \cap G \neq \varnothing$, and $F \cap \mathrm{Cl}\, W_n = \varnothing$. Since $T_n \cup \mathrm{Bd}\, W_n$ separates $X_n \cap G$ from $A_n \cap G$ and does not meet the continuum F, this is a contradiction. It follows that each subcontinuum of M which contains a nonempty open subset of I contains I.

THEOREM 2. *Suppose M is an arcwise connected continuum in S and $S-M$ does not have infinitely many complementary domains. Then* Bd M *does not contain an indecomposable continuum.*

PROOF. Assume there exists an indecomposable continuum I in Bd M. Let p be a point of an accessible composant of I. Let $U_1, U_2,$

$U_3, \cdots$ be a monotone sequence of circular regions in S centered on and converging to p. Since I has only countably many accessible composants and each composant of I is a first category subset of I, the union K of the inaccessible composants of I is a second category subset of I. Since M is arcwise connected, for each inaccessible composant C, there exists an arc A in $M-\{p\}$ which meets C and $S-C$. Let $\mathfrak{A}$ denote the set consisting of all such arcs. For each positive integer n, define L_n to be the union of all sets X such that

(1) X is a component of $(I \cup \mathrm{St}\mathfrak{A}) - U_n$ ($\mathrm{St}\mathfrak{A}$ is the union of the elements of $\mathfrak{A}$) and

(2) $X - \mathrm{Bd}\ U_n$ contains an element of $\mathfrak{A}$.

Note that for each positive integer n, each component of L_n contains a triod. Hence for each n, the components of L_n are countable. Since $K \subset \bigcup_{n=1}^{\infty}(L_n - (\mathrm{St}\ \mathfrak{A} - K))$, there exists an integer n such that L_n has a component X with the property that $X - (\mathrm{St}\mathfrak{A} - K)$ is somewhere dense in I. It follows that the closure of X with respect to M is a continuum in $M-\{p\}$ which contains an open subset of I. This is a contradiction (Theorem 1). Hence Bd M does not contain an indecomposable continuum.

THEOREM 3. *If M is an arcwise connected bounded plane continuum which does not separate the plane, then M has the fixed point property.*

PROOF. Theorem 2 and [1].

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