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THE ENDOMORPHISMS OF CERTAIN ONE-RELATOR GROUPS AND THE GENERALIZED HOPFIAN PROBLEM

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Introduction. A great deal of progress has been made in the past decade in the theory of groups with a single defining relation which possesses elements of finite order [1], [2], [3]. G. Baumslag [2] has pointed out a class $\mathcal{L}$ of torsion-free nonhopfian one-relator groups, found in [4], that support the view that torsion is a simplifying rather than a complicating factor in the theory of one-relator groups. Here we characterize the endomorphisms of the groups in $\mathcal{L}$, compute the centralizers of certain special elements and use these results to prove:

If G is in $\mathcal{L}$ then there is a proper fully invariant subgroup N of G such that G/N is isomorphic to G.

Preliminaries. $\mathcal{L}$ consists of the groups

$$G(l, m) = (a, b;\ a^{-1}b^{l}a = b^{m})$$

where $|l| \neq 1 \neq |m|$, $lm \neq 0$ and l, m are relatively prime. Let G' denote the normal closure of b in $G(l, m)$ and G'' the commutator subgroup of G'. For $n \neq 0$, let $A(n, p, q)$ denote the group

$$(X_p, \cdots, X_0, \cdots, X_q;\ X_p^{l} = X_{p+1}^{m}, \cdots, X_{q-1}^{l} = X_q^{m})$$

where $-p$ and q are maximal nonnegative integers such that $l^q | n$, $m^{-p} | n$. We then have

LEMMA 1. *The map $F: F(a) = a$, $F(b) = b$ defines an onto endomorphism of $G(l, m)$ with nontrivial kernel N where N is the normal closure of the subgroup generated by*

$$W(a, b) = ([b, a]^t b^s) b^{-1} \quad \textit{and} \quad V(a, b) = a^{-1}ba([b, a]^t b^s)^{-m}$$

such that $(m - l)t + ls = 1$.

PROOF. F is onto but not 1-1 as found in [4]. The rest is a straightforward computation.

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LEMMA 2. *$G' = (\cdots, x_i \cdots ; x_i^l = x_{i+1}^m$, i runs through the integers) and $X_i = a^i b a^{-i}$ in $G(l, m)$. G'/G'' is isomorphic to the additive subgroups $Q(l, m)$ of rationals, generated by $(l/m)^i$, under the map $H: H(x_i) = (l/m)^i$.*

LEMMA 3. *If B is in G' then B is conjugate in $G(l, m)$ to B^m iff B is conjugate in G' to x_i^k for some i, k.*

REMARK. Lemma 3 is the critical tool needed to obtain the results mentioned in the Introduction. It is proved using the techniques in [5, Chapter 4] for studying conjugacy in generalized free products with amalgamation.

LEMMA 4. *The centralizer of b^n, $n \neq 0$, in G' is $A(n, p, q)$. For $n = lk$ we have that $F(A(n, p, q)) \subseteq A(n, p, q)$.*

Main results. We now characterize the 'essential' endomorphisms T:

$$T(a) = A, \qquad T(b) = B \neq 1 \quad \text{in} \quad G(l, m)$$

and show N, of Lemma 1, is fully invariant in $G(l, m)$.

THEOREM 1. *If T is an essential endomorphism of $G(l, m)$ then*
(1) B in G', $B = D^{-1} b^k D$, D in $G(l, m)$,
(2) $DAD^{-1} = ca$, where c is in the centralizer of b^{lk} in G'.

PROOF. Note that the defining relators have a-exponent sum 0 so all relators have a-exponent sum 0. In particular $A^{-1}B^lAB^{-m}$ is a relator, so B has a-exponent sum 0 which puts B in G'. In fact G' and hence G'' are fully invariant. $A^{-1}B^lA = B^m$, so by Lemma 3, B is conjugate in G' to X_i^k and hence in G to b^k. Dividing by G'' preserves the a-exponent sum on A. Every element in G/G'' has the form ra^n where r is in $Q(l, m)$ of Lemma 2. Conjugation by a^n is seen to act in $Q(l, m)$ as multiplication by $(m/l)^n$. Letting $A \equiv ra^n \bmod G''$ and noting $b^k \equiv k \bmod G''$ reveals that when $A^{-1} BA = B^m$ is viewed mod G'' the consequence is $a^{-n}r^{-1}lk\, ra^n = mk$ and hence $(m/l)^n k = mk$. Since $B \neq 1$, $k \neq 0$, so we have $n = 1$. Thus $DAD^{-1} = ca$ where c is in G'. Now

$$(ca)^{-1} b^{lk} ca = b^{mk} = a^{-1} b^{lk} a$$

so

$$c^{-1} b^{lk} c = b^{lk}.$$

THEOREM 2. *N is fully invariant.*

PROOF. We show for any endomorphism T, $T(N) \subseteq N$. For $T(a) = A$, $T(b) = 1$, we have $T(N) = 1$. Suppose T is essential. By Theorem 1, $T(b) = D^{-1} b^k D$ and $T(a) = D^{-1} caD$ where $k \neq 0$, c in $A(lk, p, q)$.

Now $T(N)\subseteq N$ iff $D(T(W(a, b)))D^{-1}$ and $D(T(V(a, b)))D^{-1}$ are in N. Now

$$D(T(W(a, b)))D^{-1} = W(ca, b^k),$$
$$D(T(V(a, b)))D^{-1} = V(ca, b^k),$$
$$F(W(ca, b^k)) = W(F(c)a, b^{lk}),$$
$$F(V(ca, b^k)) = V(F(c)a, b^{lk}),$$

so applying Lemma 4, yields

$$W(F(c)a, b^{lk}) = 1 = V(F(c)a, b^{lk}).$$

The generalized hopfian problem. Let P be a property. G is said to be nonhopfian in the P-sense iff there is a proper normal subgroup N possessing property P such that G/N is isomorphic to G. Otherwise G is hopfian in the P-sense. The groups in $\mathfrak{L}$ are nonhopfian in the fully invariant sense. A well-known class of groups which are hopfian in the fully invariant sense are the reduced free groups.

The fully invariant subgroups of reduced free groups are the verbal subgroups [8, p. 10]. If R/V is isomorphic to R where V is verbal then the identities which generate V are identities of R so $V=1$. B. H. Neumann points out in [6, Problem 12′] that it is not known whether every reduced free group of finite rank is hopfian.

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