TRUNCATION ERROR BOUNDS FOR π-FRACTIONS

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1. **Preliminaries.** The purpose of this note is to state extensions of the results given in [2] for g-fractions. These extensions will be useful for a unification of the theory of inclusion regions for continued fractions associated with certain Hilbert transforms

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\sigma(t)}{z-t}.$$

For related results see [1], [3], and [4].

For $-\infty < a < b < +\infty$ let $W(a, b)$ denote the class of nonrational real analytic functions $f(z)$ which are holomorphic for $z \in \operatorname{comp}[a, b]$ and which satisfy $\operatorname{Re}[((z-a)(z-b))^{1/2} f(z)] > 0$ in this domain. The principal branch of the square root is assumed.

THEOREM 1. *The following alternative characterizations of the class $W(a, b)$ are valid:*

(a) *$f \in W(a, b)$ if and only if there is a bounded nondecreasing function σ, with infinitely many points of increase, such that*

$$f(z) = \int_a^b \frac{d\sigma(t)}{z-t}, \qquad z \in \operatorname{comp}[a, b];$$

(b) *$f \in W(a, b)$ if and only if f has a (unique) π-fraction expansion*

$$f(z) = \frac{\pi_0|}{|z-b} + \frac{b-a|}{|1} + \frac{\pi_1(z-a)|}{|z-b} + \frac{b-a|}{|z-b} + \frac{\pi_2(z-a)|}{|z-b} + \cdots, \qquad z \in \operatorname{comp}[a, b], \tag{1}$$

with $\pi_n > 0$, $n \geqq 0$.

2. **Inclusion regions.** The first inclusion theorem is a consequence of Theorem 1(a).

THEOREM 2. *If $f \in W(a, b)$ and z is nonreal then $f(z)$ is contained in the open convex sector $K_{-1}(z)$ bounded by the rays*

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$$\overset{b}{\kappa}_{-1}(z): w = \pi/(z-b), \quad \overset{a}{\kappa}_{-1}(z): w = \pi/(z-a), \qquad 0 \leqq \pi \leqq +\infty.$$

$K_{-1}(z)$ is precisely the set of all first approximants

$$\overset{b}{w}_1(z) = \frac{\pi'|}{|z-b} + \frac{b-a|}{|1} + \frac{\pi^*(z-a)|}{|z-b} \qquad (\pi' > 0,\ \pi^* > 0),$$

or

$$\overset{a}{w}_1(z) = \frac{\hat{\pi}|}{|z-b} + \frac{b-a|}{|1} + \frac{\tilde{\pi}(z-a)|}{|z-b} + \frac{b-a|}{|1} \quad (\hat{\pi} > 0,\ \tilde{\pi} > 0),$$

of π-fractions (1).

This result can now be extended to provide inclusion regions $K_n(z)$ which contain $f(z)$, and which are best possible if the first $n+1$ coefficients $\pi_0, \pi_1, \ldots, \pi_n$ are known. For z nonreal the linear fractional transformations

$$t_n(w) \equiv \frac{\pi_n|}{|z-b} + \frac{b-a|}{|1+(z-a)w} \qquad (n \geqq 0)$$

are nonsingular with determinants $\pi_n(b-a)(z-a)$. Let the composed transformations

$$T_n(w) \equiv t_0 \circ t_1 \circ \cdots \circ t_n(w) \qquad (T_{-1}(w) \equiv w),$$

and define

$$K_n(z) = T_n(K_{-1}(z)) \qquad (n \geqq -1).$$

The transformations T_n are also nonsingular linear fractional transformations. From Theorem 2, $K_n(z)$ is the intersection of two circular disks. Moreover the geometry of the sets $K_n(z)$ may be described completely in terms of the approximants

$$\overset{b}{w}_0(z),\ \overset{a}{w}_0(z),\ \overset{b}{w}_1(z),\ \overset{a}{w}_1(z),\ \overset{b}{w}_2(z),\ \cdots \tag{2}$$

of the π-fraction (1).

THEOREM 3. *Let z be nonreal, and let $f \in W(a, b)$ have the π-fraction expansion* (1) *with approximants* (2) *and associated sets $K_n(z)$ ($n \geqq -1$, $w^b_{-1}(z) \equiv \infty$, $w^a_{-1}(z) \equiv 0$). Then the following statements are true for $n \geqq 0$.*

(a) $f(z) \in K_n(z)$.

(b) *$K_n(z)$ is precisely the set of all $(n+2)$th approximants $w^b_{n+2}(z)$, $w^a_{n+2}(z)$ of π-fractions* (1) *with $\pi_0, \pi_1, \ldots, \pi_n$ fixed.*

(c) *$K_n(z)$ is open, bounded, and convex with interior angles*

$$\theta \equiv |\arg[(z-b)/(z-a)]|.$$

(d) $K_n(z) \subset K_{n-1}(z)$.

(e) $K_{n-1}(z) - K_n(z)$ *consists of two components* $L_n^a(z)$ *and* $L_n^b(z)$. $L_n^a(z)$ $(L_n^b(z))$ *is a circular triangle with vertices*

$$\overset{a}{w}_{n-1}(z), \quad \overset{b}{w}_n(z), \quad \overset{a}{w}_n(z) \qquad (\overset{b}{w}_{n-1}(z), \overset{a}{w}_n(z), \overset{b}{w}_n(z)),$$

and respective interior angles θ, $\alpha = |\arg(z-a)|$, $\beta = |\arg(b-z)|$.

The following limiting case of Theorem 3 is a consequence of (e).

COROLLARY. *If* $z = x > b$ *then*

$$\overset{a}{w}_0(x) < \overset{a}{w}_1(x) < \overset{a}{w}_2(x) < \cdots < \overset{b}{w}_2(x) < \overset{a}{w}_1(x) < \overset{b}{w}_0(x),$$

and if $z = x < a$ *then*

$$\overset{a}{w}_0(x) < \overset{b}{w}_1(x) < \overset{a}{w}_2(x) < \cdots < \overset{b}{w}_2(x) < \overset{a}{w}_1(x) < \overset{b}{w}_0(x).$$

3. **A priori bounds.** The theory of continued fractions, the special form of (1), and the inequality between the arithmetic and geometric means now provide bounds for $w_n^b(z) - w_n^a(z)$, and hence also for the diameter of $K_n(z)$. Furthermore special examples show that the rate of convergence implied by these bounds is best possible over the class $W(a, b)$.

LEMMA. *The function*

$$\rho(z) = \frac{(z-a)^{1/2} - (z-b)^{1/2}}{(z-a)^{1/2} + (z-b)^{1/2}} \equiv \frac{1 - \left(\dfrac{z-b}{z-a}\right)^{1/2}}{1 + \left(\dfrac{z-b}{z-a}\right)^{1/2}}$$

$$\equiv \frac{(z-a) - 2((z-a)(z-b))^{1/2} + (z-b)}{b-a}$$

maps the domain comp$[a, b]$ *conformally onto the open unit disk*: $|\rho(z)| < 1$ *for* $z \in$ comp$[a, b]$.

THEOREM 4. *For* $z \in$ comp$[a, b]$ *the diameter of* $K_n(z)$ *satisfies the inequality*

$$\operatorname{diam} K_n(z) \leqq \frac{\pi_0 |\rho(z)|^n}{|z-a||z-b|\kappa(\theta)} \qquad (n \geqq 0)$$

with

$$\kappa(\theta) \equiv \cos\frac{\theta}{2}\begin{cases}1, & 0 \leqq \theta \leqq \dfrac{\pi}{2},\\[2ex] \sin\theta, & \dfrac{\pi}{2} \leqq \theta < \pi,\end{cases} \qquad \theta \equiv \left|\arg\frac{z-b}{z-a}\right|.$$

Moreover

$$\sup_{f\in W(a,b)}\ \limsup_{n\to\infty}[\operatorname{diam} K_n(z)]^{1/n} = |\rho(z)|.$$

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