THE INEQUALITY OF SQPS AND QSP AS OPERATORS ON CLASSES OF GROUPS

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Some years ago Evelyn Nelson asked, as a special case of a question of wider interest in universal algebra, whether if $\mathfrak{X}$ is a class of groups it is always the case that SQPS $\mathfrak{X}$ = QSP $\mathfrak{X}$ (see [10], [11, Problem 3], [2] and [4, p. 161]). Here S $\mathfrak{X}$ is the class of groups isomorphic to subgroups of groups in $\mathfrak{X}$; Q $\mathfrak{X}$ is the class of groups isomorphic to factor groups of groups in $\mathfrak{X}$; and P $\mathfrak{X}$ is the class of groups isomorphic to cartesian products of families of groups in $\mathfrak{X}$. My aim is to indicate a proof that, if SL(2, q) is the group of 2 by 2 matrices of determinant 1 with entries from the field GF(q) of q elements, and $\mathfrak{X} = \{G \mid G \simeq \mathrm{SL}(2, 2^m),\ m \geqq 2\}$, then SQPS $\mathfrak{X} \neq$ QSP $\mathfrak{X}$.

The proof uses two special properties of the groups SL(2, 2^m).

Fact 1 (cf. [3, Chapter 12], or [7, Kap.II, §8]). If X is a subgroup of SL(2, 2^m) then either $X \simeq \mathrm{SL}(2, 2^l)$ for some divisor l of m, or $X \in \mathfrak{A}^2$, the class of metabelian groups. In fact, if X is not of the form SL(2, 2^l), then one knows that X is cyclic, or dihedral or a subgroup of the 1-dimensional affine group over GF(2^m), but all we shall need is that such groups are metabelian.

Fact 2. If $m \geqq 2$ then SL(2, 2^m) is simple. Moreover, there is an integer k such that for all m and all $g, h \in \mathrm{SL}(2, 2^m)$ with $g \neq 1$, h can be written as a product of exactly k conjugates of g. Here k may be taken to be 12, and the proof is a straightforward calculation.

A crucial consequence of Fact 2 is that if $X = \prod_I S_i$, where $S_i \in \mathfrak{X}$ for all $i \in I$, and if $g \in X$ then the normal closure of g is given by

$$\langle g \rangle^X = \prod \{S_i \mid i \in \mathrm{supp}(g)\} \leqq X,$$

where $\mathrm{supp}(g) = \{i \in I \mid g(i) \neq 1\}$. It follows easily that if $N \lhd X$ and $\mathfrak{E} = \{E \subseteq I \mid E = \mathrm{supp}(g) \text{ for some } g \in N\}$, then $\mathcal{E}$ is an ideal (cf. [4]) in the boolean algebra of subsets of I and $N = N_{\mathcal{E}} = \{g \in X \mid \mathrm{supp}(g) \in \mathcal{E}\}$. Therefore X/N is a reduced product (cf. [4, p. 144], or [1, p. 210]) of the groups S_i. Also

$$N = N_{\mathcal{E}} = \bigcap \{N_{\mathfrak{M}} \mid \mathcal{E} \subseteq \mathfrak{M} \text{ and } \mathfrak{M} \text{ a maximal ideal on } I\}.$$

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Thus, since $X/N_{\mathfrak{M}}$ is an ultraproduct ([**1**, p. 210], [**4**, p. 145]) of the groups S_i, we see that X/N is residually an ultraproduct of the groups in $\mathfrak{X}$.

Now, from Fact 1, s $\mathfrak{X}\subseteq\mathfrak{X}\cup\mathfrak{A}^2$ and so, if $G\in$ PS $\mathfrak{X}$ then $G=X\times Y$ where X is a cartesian product of groups in $\mathfrak{X}$, and Y is metabelian. If $N\lhd G$ then it follows from Fact 2 that $N=(X\cap N)\times(Y\cap N)$ and $G/N\simeq(X/X\cap N)\times(Y/Y\cap N)$. Thus if $H\in$ QPS $\mathfrak{X}$ then $H=X\times Y$, where X is residually an ultraproduct of groups in $\mathfrak{X}$ and Y is metabelian. Since $\mathfrak{X}$ consists of 2-dimensional linear groups, an ultraproduct of groups in $\mathfrak{X}$ is a 2-dimensional linear group over a suitable field (see, for example, Kegel [**8**]). Furthermore, a linear group is locally residually finite (Mal'cev [**9**, Theorems VII, VIII]). Therefore the direct factor X of H is residually locally residually finite, that is X is locally residually finite. Since, by a theorem of P. Hall [**5**], Y is also locally residually finite, it follows that H, and then every subgroup of H has this property. That is, SQPS $\mathfrak{X}$ consists of locally residually finite groups.

On the other hand, QSP $\mathfrak{X}$ is the variety generated by $\mathfrak{X}$, and this is known [**6**, p. 45] to be the variety $\mathfrak{O}$ of all groups. Since there is no dearth of groups, such as the finitely generated infinite simple groups, or finitely generated non-Hopf groups, or the group of permutations of the integers generated by the infinite cycle $(\cdots, -1, 0, 1, 2, 3, \cdots)$ and the 3-cycle $(1, 2, 3)$, which are not locally residually finite, it follows that SQPS $\mathfrak{X}\neq$ QSP $\mathfrak{X}$, as promised.

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