ON THE AVERAGE ORDER OF SOME ARITHMETICAL FUNCTIONS

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ABSTRACT. We consider a large class of arithmetical functions generated by Dirichlet series satisfying a functional equation with gamma factors. Our objective is to state some Ω results for the average order of these arithmetical functions.

Our objective here is to state some Ω-theorems on the average order of a class of arithmetical functions.

We indicate very briefly the class of arithmetical functions under consideration. For a more complete description, see [4].

Let $\{a(n)\}$ and $\{b(n)\}$ be two sequences of complex numbers, not identically zero. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two strictly increasing sequences of positive numbers tending to ∞. Put $s=\sigma+it$ with σ and t both real and suppose that

$$\phi(s) = \sum_{n=1}^{\infty} a(n)\lambda_n^{-s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} b(n)\mu_n^{-s}$$

each converge in some half-plane. Let σ_a^* denote the abscissa of absolute convergence of ψ. Put

$$\Delta(s) = \prod_{\nu=1}^{N} \Gamma(\alpha_\nu s + \beta_\nu),$$

where $\alpha_\nu > 0$ and β_ν is complex, $\nu = 1, \cdots, N$. Assume that for sone real number r, ϕ and ψ satisfy the functional equation $\Delta(s)\phi(s) = \Delta(r-s)\psi(r-s)$.

We shall consider the Riesz sum

$$A_q(x) = \frac{1}{\Gamma(q+1)} \sum_{\lambda_n \leq x} a(n)(x-\lambda_n)^q,$$

where $q \geqq 0$. Let $\alpha = \sum_{\nu=1}^{N} \alpha_\nu$ and define

$$Q_q(x) = \frac{1}{2\pi i} \int_{C_q} \frac{\Gamma(s)\phi(s)x^{s+q}}{\Gamma(s+q+1)}\, ds,$$

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where C_q is a cycle encircling all of the singularities of the integrand to the right of $\sigma=-q-1-k$, where $k>|\frac{1}{2}r-1/(4\alpha)|$, and where all of the singularities of ϕ lie in $\sigma>-k$. Then, the "error term" $P_q(x)$ is defined by

$$P_q(x) = A_q(x) - Q_q(x).$$

Furthermore, let

$$\beta(q) = \beta = -\sum_{\nu=1}^{N} \beta_\nu + \tfrac{1}{2}N - \tfrac{1}{2}r\alpha - \tfrac{3}{4} - \tfrac{1}{2}q,$$

$$\theta(q) = \theta = \tfrac{1}{2}r - 1/(4\alpha) + q - q/(2\alpha),$$

and

$$\kappa(q) = \kappa = \sigma_a^* - \tfrac{1}{2}r - 1/(4\alpha) - q/(2\alpha).$$

From [4, p. 111], $\kappa(0)\geqq 0$. In the sequel we assume that $\kappa(q)\geqq 0$. If $\kappa(q)<0$, the order of $P_q(x)$ can be determined exactly [4, Theorem 3.2].

We are now ready to state

THEOREM 1. *Assume that* $b(n)\geqq 0$ *and that* β_ν *is real,* $\nu=1, \cdots, N$. *Suppose that there exist constants* c *and* ρ *such that as* x *tends to* ∞,

$$\sum_{\mu_n\leqq x} b(n) \sim cx^{\sigma_a^*}\log^{\rho-1} x.$$

Lastly, suppose that $\mu_{n+1}-\mu_n=o(\mu_n)$, *as* n *tends to* ∞. *Then, if* $\cos(\beta\pi)>0$ *and* $\kappa>0$,

$$\operatorname{Re}\{P_q(x)\} = \Omega_+(x^\theta\{\log x\}^\kappa\{\log\log x\}^{\rho-1});$$

if $\cos(\beta\pi)<0$ *and* $\kappa>0$,

$$\operatorname{Re}\{P_q(x)\} = \Omega_-(x^\theta\{\log x\}^\kappa\{\log\log x\}^{\rho-1}).$$

The proof of Theorem 1 for $q=0$ is given in [1]. The proof of the more general theorem given here follows along the same lines. The idea of the proof goes back to Szegö [7] and Szegö and Walfisz [8]. Dirichlet's theorem on the simultaneous approximation of a finite set of real numbers is used in the proof, and it is at this stage of the proof that the restriction $b(n)\geqq 0$ is necessary.

Results of Hardy [5] on $r_2(n)$, the number of representations of n as the sum of two squares, and on $d(n)$, the divisor function, are special cases of Theorem 1. Results of Szegö [7] on $r_k(n)$ and Szegö and Walfisz [8] on the Piltz divisor problem in algebraic number fields are also special cases.

For the arithmetical functions under consideration, Theorem 1 is an improvement upon general theorems of Landau [6] and Chandrasekharan and Narasimhan [3], [4].

Theorem 1 yields only "one-sided" results. In many cases, however, we can obtain "two-sided" results as the following theorem shows.

THEOREM 2. *Assume the hypotheses of Theorem* 1. *Let* $Q_\psi(x)$ *be* $Q_0(x)$ *except that* ϕ *is replaced by* ψ. *Suppose that as* x *tends to* ∞,

$$Q_\psi'(x) \sim c\sigma_a^* x^{\sigma_a^*-1} \log^{\rho-1} x.$$

Let $\gamma(q)=\gamma=2\alpha\kappa-1$, *and for* $\kappa>0$ *and* a *real define*

$$g(a) = \int_0^\infty e^{-u^2} u^\gamma \cos(au+\beta\pi)du.$$

Then, if $\kappa>0$ *and* $g(a)$ *has a change in sign,*

$$\text{(1)} \qquad \operatorname{Re}\{P_q(x)\} = \Omega_\pm(x^\theta\{\log x\}^\kappa\{\log\log x\}^{\rho-1});$$

if $\kappa=0$, *in all cases,*

$$\text{(2)} \qquad \operatorname{Re}\{P_q(x)\} = \Omega_\pm(x^\theta\{\log\log x\}^\rho).$$

The assumption in Theorem 1 that $\cos(\beta\pi)\neq 0$ has been removed. However, we have an additional restriction in that $g(a)$ has a change in sign. In [2] we establish some general conditions under which $g(a)$ has a sign change. We also determine there some conditions under which $g(a)$ has no sign change. It is very unfortunate, indeed, that the most interesting cases of $r_2(n)$ and $d(n)$ for $q=0$ fall into this latter category.

The proof of Theorem 2 for $q=0$ and $\kappa>0$ is given in [2], and the proof for $q>0$ follows along the same lines. For $\kappa=0$, the proof is, in fact, somewhat easier. The idea for the proof of Theorem 2 goes back to Szegö and Walfisz [9], and so their results on the Piltz divisor problem for algebraic number fields are special cases of Theorem 2. Again, Dirichlet's theorem is used in the proof, but in a different way, however.

Our next theorem yields some information on how often the inequalities (1) and (2) in Theorem 2 are valid.

THEOREM 3. *Assume the hypotheses of Theorem* 2. *Then, there exist positive constants* c_1 *and* c_2 *and a positive, strictly increasing sequence* $\{y_n\}$ *tending to* ∞ *such that both inequalities*

$$\pm\operatorname{Re}\{P_q(x)\} > c_1 x^\theta(\log x)^\kappa(\log\log x)^{\rho-1}$$

if $\kappa > 0$, *and*

$$\pm \operatorname{Re}\{P_q(x)\} > c_1 x^\theta (\log \log x)^\rho$$

if $\kappa = 0$, *have solutions in each interval*

$$y_n \leqq x \leqq y_n + c_2 y_n^{1-1/(2\alpha)} (\log y_n)^{1/2-1/(2\alpha)} .$$

For $\kappa > 0$ and $q = 0$ the proof is given in [2]. The proof of the more general Theorem 3 is exactly the same. Theorem 3 gives an improvement upon a general theorem of Landau [6] for the arithmetical functions under consideration.

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