RELATIVISTIC COVARIANCE OF AN INTERACTING QUANTUM FIELD

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I. **Introduction.** Physicists believe that quantum field theory can describe the interactions between elementary particles. The difficulties in constructing model quantum field theories become more tractable in two dimensional space-time. The best understood two dimensional model is the "$\lambda(\phi^4)_2$ quantum field theory" [1]–[4]. We discuss this model which describes a self interacting boson field $\phi(x, t)$.

Let $\mathcal{B}$ be a bounded open subset of R^2. For $(x, t) \in \mathcal{B}$, the field $\phi(x, t)$ is a sesquilinear form defined on a dense domain $\mathfrak{D}$ in a Hilbert space $\mathcal{H}$. Furthermore ϕ is continuous in (x, t) and satisfies the nonlinear partial differential equation

$$\left\{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right\}\phi + 4\lambda\phi^3 = 0. \tag{1}$$

The nonlinear term $(\phi^3)(x, t)$ is defined in [3], and (1) holds on $\mathfrak{D}\times\mathfrak{D}$ as an equation for Schwartz distributions.

For real $f \in \mathcal{C}_0^\infty$, the sesquilinear form

$$\phi(f) = \int \phi(x, t)f(x, t)dx\,dt \tag{2}$$

uniquely determines a selfadjoint operator $\phi(f)$ [3]. Let $\mathfrak{A}(\mathcal{B})$ denote the von Neumann algebra

$$\mathfrak{A}(\mathcal{B}) = \{e^{i\phi(f)}: f = \bar{f} \in \mathcal{C}_0^\infty, \operatorname{supp} f \subset \mathcal{B}\}''. \tag{3}$$

One can interpret $\mathfrak{A}(\mathcal{B})$ as the bounded observables in the space-time region $\mathcal{B}$. It is convenient to work with the C^*-algebra $\mathfrak{A}$ of quasilocal observables defined as the norm closure of $\bigcup_{\mathcal{B}\subset R^2}\mathfrak{A}(B)$.

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The field $\phi(x, t)$ is also space time translation covariant [3]–[4]. If $a=(\alpha, \tau)\in R^2$, there is a unitary operator

$$U(a) = \exp(iH\tau - iP\alpha) \tag{4}$$

such that

$$\phi(x+\alpha, t+\tau) = U(a)\phi(x, t)U(a)^*. \tag{5}$$

The transformation (4)–(5) uniquely determines a *-isomorphism

$$\sigma_a : \mathfrak{A}(\mathcal{B}) \to \mathfrak{A}(\mathcal{B}+a) = U(a)\mathfrak{A}(\mathcal{B})U(a)^* \tag{6}$$

for each $\mathcal{B}$, and σ_a extends uniquely to a *-automorphism of $\mathfrak{A}$.

II. **Relativistic covariance.** Covariance of the field under Lorentz transformation is a usual axiom of quantum field theory. We prove that this axiom holds for the $\lambda(\phi^4)_2$ theory of [1]–[4]. To formulate Lorentz covariance, let $\mathcal{P}$ denote the restricted Poincaré group of transformations. An element $\{a, \Lambda_\beta\}\in\mathcal{P}$ is defined by its action on R^2,

$$\{a, \Lambda_\beta\}(x, t) = (x(\cosh\beta) + t(\sinh\beta) + \alpha, \qquad x(\sinh\beta) + t(\cosh\beta) + \tau). \tag{7}$$

Hence $\mathcal{P}$ is the semidirect product of the space-time translations a with the pure Lorentz transformations Λ_β (corresponding to velocity boosts $\tanh\beta$).

THEOREM. *For each $\{a, \Lambda_\beta\}\in\mathcal{P}$ and each bounded set $\mathcal{B}\subset R^2$, there is a unitary operator U so that for all $(x, t)\in\mathcal{B}$,*

$$\phi(\{a, \Lambda_\beta\}(x, t)) = U\phi(x, t)U^*, \tag{8}$$

as an equality between operator valued distributions.

COROLLARY 1. *There is a representation $\sigma_{\{a, \Lambda_\beta\}}$ of $\mathcal{P}$ by *-automorphisms of $\mathfrak{A}$ such that*

$$\sigma_{\{a, \Lambda_\beta\}} : \mathfrak{A}(\mathcal{B}) \to \mathfrak{A}(\{a, \Lambda_\beta\}\mathcal{B}). \tag{9}$$

We first establish the theorem for pure Lorentz transformations $\{0, \Lambda_\beta\}$ and compact sets $\mathcal{B}_1$ in the quadrant $\{x>|t|\}$. Given $\beta_0>0$, we construct a unitary group $U(\beta)=\exp(iM\beta)$ satisfying (8) for $|\beta|<\beta_0$. Our choice of a selfadjoint local generator M is motivated by familiar formal expressions in the physics literature. It is technically convenient to have $0\leqq M$, which is possible only if $\mathcal{B}_1\subset\{x>|t|\}$.

The key step in proving (8) is to show that $\phi(x, t)$ of (5) satisfies the partial differential equation

$$\left\{t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}\right\}\phi(x, t) = [iM, \phi(x, t)]. \tag{10}$$

We obtain (10) from a lengthy series of estimates which allow us to compute $[iM, \phi]$ on a dense subset of $\mathcal{H}\times\mathcal{H}$. We then integrate (10) to obtain (8) for our special case. The general case follows, since a Poincaré transformation from an arbitrary bounded set $\mathfrak{B}\subset R^2$ can be expressed as a product of space-time translations (6) and a pure Lorentz transformation from a compact set $\mathfrak{B}_1\subset\{x>|t|\}$.

Corollary 1, together with the results of [3] asserts the following:

COROLLARY 2. *The algebras $\mathfrak{A}(\mathfrak{B})$ and $\mathfrak{A}$ of the $\lambda(\phi^4)_2$ quantum field theory satisfy all of the Haag-Kastler axioms for a quantum field theory* [5].

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