LIE ALGEBRAS OF ANALYTIC VECTOR FIELDS AND UNIQUENESS IN THE CAUCHY PROBLEM FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS[1]

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Let $P(x, D)$ be a partial differential operator defined in an open set $\Omega \subset R^n$ and let $x^0 \in \Omega$ be a boundary point of a closed subset F of Ω. We say that there is uniqueness in the Cauchy problem (UCP) for the system (P, x^0, F) if to every open neighborhood $U \subset \Omega$ of x^0 there is an open neighborhood $V \subset U$ of x^0 such that for every distribution u in U,

$$P(x, D)u = 0 \quad \text{in } U, \text{supp } u \subset F \cap U \quad \Rightarrow u = 0 \quad \text{in } V.$$

The classical uniqueness theorem of Holmgren (as extended to distribution solutions by Hörmander [1]) gives a sufficient condition for UCP for the system (P, x^0, F) in the case in which P is a linear partial differential operator with analytic coefficients and the boundary of F is a C^1 hypersurface S. This condition is that S is not characteristic with respect to P at x^0. Although this condition is sufficient for UCP it is certainly not necessary. Malgrange [2], Hörmander [1], Trèves [3] and Zachmanoglou [4], [5], [6] have obtained some necessary and some sufficient conditions for UCP but the general problem is still unsolved.

In this note we present a necessary and sufficient condition for UCP for first order linear partial differential operators with analytic complex valued coefficients. No additional assumptions on the closed set F are made.

Let $\mathfrak{A}$ denote the ring of all real-valued analytic functions in Ω and let

$$(1) \quad P(x, D) = A + iB + c(x) = \sum_{j=1}^{n} a^j(x) D_j + i \sum_{j=1}^{n} b^j(x) D_j + c(x),$$

where $a^1, \cdots, a^n, b^1, \cdots, b^n$, Re c and Im c belong to $\mathfrak{A}$, $i = \sqrt{-1}$ and $D_j = \partial/\partial x_j$. A and B can be thought of as vector fields with coefficients in $\mathfrak{A}$. A trajectory of a collection $\mathfrak{C}$ of analytic vector fields is

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a piecewise analytic curve each analytic piece of which is an integral curve of an element of $\mathcal{C}$. Let

$$\mathcal{A}(A, B) = \{\alpha A + \beta B : \alpha, \beta \in \mathcal{A}\}.$$

The following theorem asserts that the zeroes of solutions of $P(x, D)u = 0$ propagate along trajectories of $\mathcal{A}(A, B)$.

THEOREM 1. *For any distribution u in Ω and any open subset Ω_0 of Ω,*

$$\begin{matrix} P(x, D)u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega_0 \end{matrix} \Rightarrow u = 0 \quad \text{in } \tilde{\Omega}_0(P, \Omega)$$

where $\tilde{\Omega}_0(P, \Omega)$ is the set of points of Ω which can be connected to points of Ω_0 by trajectories of $\mathcal{A}(A, B)$ contained in Ω.

The proof is based on a general theorem concerning the propagation of zeroes of solutions of linear partial differential equations with flat characteristic cones [7].

Let $\mathfrak{M}^{x^0}(\mathcal{A}(A, B))$ denote the set of points in Ω which can be connected to x^0 by trajectories of $\mathcal{A}(A, B)$ contained in Ω. It is easy to see that Theorem 1 implies that the following condition is sufficient for UCP for the system (P, x^0, F): $\mathfrak{M}^{x^0}(\mathcal{A}(A, B))$ intersects the complement of the set F in every neighborhood of x^0. Thus it becomes necessary to study closely the set $\mathfrak{M}^{x^0}(\mathcal{A}(A, B))$, at least in some neighborhood of the point x^0. It turns out that the nature of this set depends on the Lie algebra generated by the vector fields A and B.

The bracket of two analytic vector fields is defined by $[A, B] = AB - BA$ and it is also an analytic vector field. The bracket operation has certain well-known properties which will not be mentioned here. The Lie algebra generated by A and B is denoted by $\mathcal{L}(A, B)$ and is defined as the set of all linear combinations with coefficients in $\mathcal{A}$ of A, B and all vector fields obtained by repeated application of the bracket operation on A and B. By $\dim \mathcal{L}(A, B)|_{x=x^0}$ we denote the dimension of the vector space obtained from $\mathcal{L}(A, B)$ by evaluating the coefficients at x^0. Clearly $\dim \mathcal{L}(A, B)|_{x=x^0}$ may vary from point to point in Ω but we always have

$$0 \leq \dim \mathcal{L}(A, B)|_{x=x^0} \leq n.$$

The analyticity of the vector fields A and B implies the following interesting theorem.

THEOREM 2. *Let x^0 be any point in Ω and suppose that*

$$\dim \mathcal{L}(A, B)|_{x=x^0} = k,$$

where $0 \leq k \leq n$. Then there is an open neighborhood $U \subset \Omega$ of x^0 and a

k-dimensional manifold $\mathfrak{M}^{x^0}(A, B)$ passing through x^0 and contained in U and such that at every point of $\mathfrak{M}^{x^0}(A, B)$,

(i) dim $\mathfrak{L}(A, B) = k$,

(ii) *every element of $\mathfrak{L}(A, B)$ is interior (tangent) to $\mathfrak{M}^{x^0}(A, B)$.*

When $k=0$ or $k=n$ the conclusions of the theorem are immediate and do not depend on the analyticity of A and B. However when $1 \leqq k < n$ the assumption of analyticity is essential. When $k \geqq 1$ we may assume that $A|_{x=x^0} \neq 0$ and the theorem is proved by showing that there is an analytic transformation of coordinates such that in the new coordinates, having origin corresponding to x^0 and in some neighborhood of the origin, A and B have the form,

$$\begin{aligned} A &= D_1, \qquad B = B_{(k)} + B^{(k)}, \\ B_{(k)} &= B_{(k)}(x, D_{(k)}) = b^1(x)D_1 + \cdots + b^k(x)D_k, \\ B^{(k)} &= B^{(k)}(x, D^{(k)}) = b^{k+1}(x)D_{k+1} + \cdots + b^n(x)D_n, \end{aligned} \tag{3}$$

where

$$B^{(k)}(x, D^{(k)})\,|_{x^{(k)}=0} = 0, \tag{4}$$

$$\dim \mathfrak{L}(A, B_{(k)})\,|_{x^{(k)}=0} = k \tag{5}$$

and, moreover,

$$\mathfrak{L}(A, B)\,|_{x^{(k)}=0} = \mathfrak{L}(A, B_{(k)})\,|_{x^{(k)}=0}. \tag{6}$$

Here we use the notation $x_{(k)} = (x_1, \cdots, x_k)$, $D_{(k)} = (D_1, \cdots, D_k)$, $x^{(k)} = (x_{k+1}, \cdots, x_n)$, $D^{(k)} = (D_{k+1}, \cdots, D_n)$. Note that the equation $x^{(k)} = 0$ defines the k-dimensional manifold $\mathfrak{M}^{x^0}(A, B)$.

At the time of the typing of this announcement it was brought to the attention of the author that Theorem 2 is a special case of a general theorem on Lie algebras of analytic vector fields on an analytic manifold, published in 1966 by Nagano [8].

Let $\mathfrak{M}^{x^0}(\mathfrak{L}(A, B))$ denote the set of points in Ω which can be connected to x^0 by trajectories of $\mathfrak{L}(A, B)$ contained in Ω. In view of Theorem 2 the following theorem is immediate and it provides a means for constructing the manifold $\mathfrak{M}^{x^0}(A, B)$ by solving ordinary differential equations.

THEOREM 3. *In some neighborhood of x^0,*

$$\mathfrak{M}^{x^0}(A, B) = \mathfrak{M}^{x^0}(\mathfrak{L}(A, B)).$$

The following theorem leads us back to our original problem of uniqueness in the Cauchy problem.

THEOREM 4. *In some neighborhood of* x^0,

$$\mathfrak{M}^{x^0}(A, B) = \mathfrak{M}^{x^0}(\mathfrak{A}(A, B)).$$

In view of Theorem 2 it is enough to show that if $\dim \mathfrak{L}(A, B)|_{x=x^0} = n$ then every point in some neighborhood of x^0 can be connected to x^0 by a trajectory of $\mathfrak{A}(A, B)$ contained in that neighborhood.

Let us denote the manifold described in Theorems 2, 3 and 4 by $\mathfrak{M}(P, x^0)$. In view of Theorems 1 and 4 we will call $\mathfrak{M}(P, x^0)$ the zero propagator of $P(x, D)$ at x^0,

$$\mathfrak{M}(P, x^0) = \mathfrak{M}^{x^0}(A, B) = \mathfrak{M}^{x^0}(\mathfrak{L}(A, B)) = \mathfrak{M}^{x^0}(\mathfrak{A}(A, B)).$$

In the language of differential geometry, $\mathfrak{M}(P, x^0)$ is the maximal integral manifold passing through x^0 of the Lie subalgebra of analytic vector fields on Ω generated by the real and imaginary parts of the principal part of $P(x, D)$. Now, combining Theorems 1 and 4 we obtain a sufficient condition for UCP.

THEOREM 5. *Let* Ω *be an open set in* R^n, $P(x, D)$ *a linear first order partial differential operator with analytic complex-valued coefficients in* Ω *and* $x^0 \in \Omega$ *a boundary point of a closed subset* F *of* Ω. *There is uniqueness in the Cauchy problem for the system* (P, x^0, F) *if for every open neighborhood* $U \subset \Omega$ *of* x^0,

$$\mathfrak{M}(P, x^0) \cap (U \sim F) \neq \emptyset, \tag{7}$$

i.e. the zero propagator of $P(x, D)$ *at* x^0 *intersects the complement of* F *in every neighborhood of* x^0.

COROLLARY. *If* $\dim \mathfrak{L}(A, B)|_{x=x^0} = n$ *then there is always uniqueness in the Cauchy problem for the system* (P, x^0, F) *for any closed set* F.

Thus, if at each point of an open set $\Omega \subset R^n$, $\dim \mathfrak{L}(A, B) = n$ then the zeroes of solutions of the equation $P(x, D)u = 0$ propagate in exactly the same way as those of elliptic equations: For any open subset Ω_0 of Ω and any distribution u in Ω, the conditions $P(x, D)u = 0$ in Ω and $u = 0$ in Ω_0 imply that $u = 0$ in every connected component of Ω which intersects Ω_0.

THEOREM 6. *If the principal part of* $P(x, D)$ *does not vanish at* x^0 *then condition* (7) *is also necessary for uniqueness in the Cauchy problem for the system* (P, x^0, F).

Theorem 6 is proved using formulas (3) and (4) and showing that there is a solution u of $P(x, D)u = 0$ in some open neighborhood of x^0 such that supp $u = \mathfrak{M}(P, x^0)$.

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