BOOK REVIEWS

The algebraic theory of semigroups. Vol. I and II by A. H. Clifford and G. B. Preston. Mathematical Surveys, number 7, American Mathematical Society, Providence, Rhode Island, 1961 and 1967. xvi+244 pp. and xv+350 pp. $11.10 and $14.20.

Semigroups by E. S. Ljapin. Fizmatigiz, Moscow, 1960. vii+447 pp; English translation, Translations of Mathematical Monographs, volume 3, American Mathematical Society, Providence, Rhode Island, 1963. x+487 pp. $22.20.

The theory of finitely generated commutative semigroups by László Rédei. Translated, edited by N. Reilly, Pergamon Press, New York, 1965, xiii+353 pp. $13.50.

Elements of compact semigroups by Karl Heinrich Hofmann and Paul S. Mostert. Charles E. Merrill Books, Inc., Columbus, Ohio, 1966, xiii+384 pp. $15.00.

A semigroup is a set with an associative multiplication. No more and no less.

Recent books on semigroups seem either to attempt to present a survey of the existing theory, such as it is, or to place in one package the author's new research contributions.

Clifford and Preston's two volumes succeed admirably in presenting the algebraic (nontopological and nonordered) theory of semigroups developed up to around 1964. Chapter 1 contains some elementary definitions and theorems, e.g., the structure of cyclic semigroups, that maximal subgroups are disjoint and exist if there are idempotents, and the useful, elementary Ore-Dubreil condition for the embedding of noncommutative semigroups in groups with the elegant proof by Rees.

The next two chapters form the heart of the work. Following Green's 1951 Annals paper, one thinks of the semigroup as the multiplicative part of a ring, and calls two elements L (resp. R, resp. J) equivalent iff they generate the same principal left (resp. right, resp. 2-sided) ideal. These Green equivalence relations are defined for all semigroups. They also define $H=L\cap R$ and $D=L\circ R=R\circ L$. In Chapter 2 Clifford and Preston introduce the Green correspondence between D equivalent H-classes and also show that every maximal subgroup is an H-class and conversely each H-class containing an idempotent is a maximal subgroup. Also an analysis of the D-classes containing an idempotent (so-called regular D-classes) is carried out and two D equivalent maximal subgroups are shown to be isomorphic.

For an arbitrary H-class the Schützenberger group is introduced which is the left idealizer of the H-class considered as acting as regular permutations on the H-class. When the H-class H is a group the Schützenberger group equals H.

If I is a (2-sided) ideal of S, then S/I is the Rees quotient given by pinching I to 0 giving $(S-I)\cup 0$ with the natural multiplication. Then, as every algebraist would instinctively ask, what is $F=I_1/I_2$ if I_1 and I_2 are ideals of S and I_2 is proper but maximal in I_1? Not surprisingly, the first answer supplied in Chapter 2 is that either F is null ($F^2=0$) or that $F^2=F$ and the only ideals of F are 0 and F, (0-simple), 'ust as in ring theory. Also $I_2-I_1=J$ is a J-class of S and F is just J^0 with the product defined to be zero when not in J, so F determines the local multiplication.

This sets the stage for a "Wedderburn Type Theorem and Proof" which would classify the 0-simple semigroups. Around 1940 Philip Hall suggested to D. Rees that he attack this problem for a dissertation. The successful results are brought together in Chapter 3. Assuming that the 0-simple semigroup S has a primitive idempotent (equivalently there exists 0-minimal left or right ideals), Rees shows that S is isomorphic to the following kind of semigroup: Let A and B be sets and G a group. Let C be a $B\times A$ matrix with coefficients in G or zero and assume C is nonzero at least once in each row and column. Then consider the semigroup of all $A\times B$ matrices with at most one nonzero entry in the entire matrix. Finally multiply two such matrices X and Y by $X\cdot Y=XCY$. The proof presented uses the results developed previously concerning this structure of regular D-classes and the Green relations. In fact it is shown that S 0-simple has only one nonzero $J=D$ class and A is chosen to be nonzero R classes, B the nonzero L classes, and G an H-class containing an idempotent. The converse is trivial.

If one listens closely at this stage of the book one can hear trumpets blowing as the Rees Theorem is proved.

Let D be a D-class of S. Then S acts on the right of D^0 by right multiplication. By using the previous results, the authors show that the action of an element gives rise to a right translation of D^0 which can be represented by a $B\times B$ row monomial matrix with coefficients in G^0. The resulting homomorphism of S into row monomial matrices is called the right Schützenberger representation of S over D. Chapter 3 ends with a proof that if S is regular, then the direct sum of all the left and right Schützenberger representations with respect to all the D classes is faithful, a result originally due to Schützenberger and Preston.

Chapter 4 shows that a semigroup is a union of groups iff $s \rightarrow S^1 s S^1$ is a homomorphism of S into the set of subsets of S under intersection, a result originally due to Clifford. The decomposition of a commutative semigroup into its Archimedean components is also proved.

Chapter 5 reviews the Wedderburn Theory for finite dimensional algebras and proves Munn's result, which determines the irreducible representations of S, assuming S satisfies DCC for principal ideals. The essential idea is that a nonzero irreducible representation of an ideal I of S has a unique extension to an irreducible representation of S. Thus the irreducible representation is determined by its values on the 'first' J-class J on which it is nonzero. J must be regular. If Rees' Theorem applies to J, then the semigroup algebra of J^0 can be explicitly computed. This is done in the latter part of Chapter 5. Unfortunately the authors choose to hack it out with matrix calculations, instead of applying Wedderburn to G and then bringing the structure matrix to diagonal form with ones and zeros down the diagonal which then gives the ring a transparent structure from which the results could be read off. The assumptions needed for all this are clearly satisfied if S is a finite semigroup. In this case the theory easily yields nasc for the semigroup algebra to be semisimple (all representations completely reducible). Finally the chapter closes by abruptly ignoring the previous results and proving some theorems on the characters of commutative semigroups (which contain a few minor errors). No character theory for noncommutative semigroups is presented. So ends Volume I. All in all a success.

The high point of the second volume by Clifford and Preston, which appeared about six years after the first, is the last chapter which gives nasc for the embeddability of a semigroup in a group. The authors give Malcev's example of a cancellative semigroup which is not embeddable in a group and go ahead and prove the Malcev nasc conditions for embedding. Though theoretically important, the conditions are countably infinite in number and no nonfinite subset will suffice. They also exposit the geometric polyhedral condition of Lambek's which is also nasc and establish the relation between the Malcev and Lambek conditions. The technical device used is the so-called free group on S which is the functorially maximal way of mapping S homomorphically into the group G so the image is a set of group generators. Then clearly S is embeddable into a group iff this works. Since things work nicely under restriction they easily prove that S is embeddable iff every finitely generated subsemigroup is embeddable. Now by examining the free group over S they derive the Ptak sufficient condition and the results noted above. The style of this chapter reminds one of the papers of Boone on the word problems for groups.

Another excellent chapter of the second volume is Chapter 9, entitled "Finite presentations of semigroups and free products with amalgamation." This chapter starts with conditions that subsemigroups of free semigroups are free and that every semigroup can be embedded in a two generated semigroup. Then the authors take up Rédei's theory of finitely generated commutative semigroups. The main result is that all finitely generated commutative semigroups are finitely presented. In particular there are only a countable number of them. They present this result using Rédei's kernel functions. Those readers following Clifford and Preston's clear presentation can easily prove that all finitely generated commutative semigroups satisfy the DCC on submorphisms. Thus in fourteen pages of this chapter the authors cover the material which Rédei covers in 353 pages of his book! Rédei's research is interesting, but in the writing of this monograph his enthusiasm ran away with him while he performs the amazing feat of creating a 353 page book from a 30 page research article. Chapter 9 ends with Howie's results on sufficient conditions for embedding two semigroups with amalgamations, using free products with amalgamations, and Croisot's results on constructing semigroups with cancellation.

Chapter 6 of Volume II discusses semigroups containing 0-minimal ideals and at the end it studies the various chain conditions on principal left, right 2-sided ideals.

Chapter 8 studies simple (no proper ideals) and 0-simple semigroups. When Rees' Theorem applies, the structure is known, but Bruck's Theorem asserts that any semigroup can be embedded in a simple semigroup with an identity (which is proved in eleven lines). The authors give a survey of certain classes of simple semigroups.

The remaining three chapters of the second volume entitled "Inverse semigroups, congruences, and representations by transformation on a set" are not up to the level of the rest of the other nine chapters and thus are disappointing. Chapter 11 could have been made a problem. The results on congruences of Chapter 10 could be easily deduced from the previous material of Volume I or are naive generalizations of the homomorphism theorem of group theory. The applications to inverse semigroups could have been done directly.

The great strength of the two volumes is that it is the first reasonable exposition of the algebraic theory of semigroups. Its weakness is that it presents no point of view. It perhaps would have been better to have made one volume by omitting Chapters 7 and 10 and condensing elsewhere in the second volume. The style is somewhat overly pedantic, especially in chapters having very few results of any depth, where things are constantly overproved. The method and style of credits bends over backwards to be fair and scrupulous e.g., on p. 159 A · · · is credited with noting that the ring of polynomials in x is the semigroup algebra over the infiite cyclic semigroup. (I looked for; but did not find a footnote giving Gutenberg credit for the type.)

The time was ripe for a systematic exposition of the algebraic theory of semigroups, and Clifford and Preston succeeded in two volumes. But one volume might have sufficed, for what the field needed was insight and direction, and this they left to others.

The book *Semigroups* by E. S. Ljapin attempts to cover the same ground as Clifford and Preston and the most charitable comment we can make is that the reader would be better off reading Clifford and Preston.

Elements of compact semigroups by Hofmann and Mostert is a good book. From its pun in the title to its ghoulish pictures and Cantorian swastikas at the end it has a good plot (an old good one redressed) and a good finish.

The authors wisely assume Clifford and Preston as pre-requisite. The first Chapter A is preliminaries. They review minimal ideals, homomorphisms, ideals, Schützenberger groups and regular D-classes, Rees semigroups and Clifford semigroups (=union of groups) supplementing the standard algebraic treatment by the necessary topological trappings. They review the semigroup of compact sets and projective limits.

Chapter A closes with the following theorem due to A. H. Wallace. Let S be a compact semigroup. Then there is a unique compact minimal ideal $M(S)$. Also Rees' Theorem applies to $M(S)$ topologized in the obvious way. Further if S is connected and has an identity and H is a maximal subgroup of $M(S)$, then H is compact and the inclusion map induces an isomorphism on the Čech or Alexander-Spanier cohomology over any coefficient group. This is easily seen by applying the generalized homotopy theorem. Thus in the connected compact monoid case the space A and B from Rees' Theorem and the group H are acyclic and so connected.

The first section of second Chapter B characterizes monothetic (=compact dense cyclic semigroup) and solenoidal (=compact dense one-parameter semigroup). This is done by constructing the universal example of which all others are surmorphism images and then more or less classifying the surmorphisms. They use Pontrjagin duality to solve the problem for groups and then "wrap a tail." The duality results of Pontrjagin are summarized in an appendix with standard references for the hard core proofs. In the next section the authors also consider so-called cylindrical semigroups which are surmorphisms of the universal solenoidal semigroup direct sum a compact group. The next section, 3, of Chapter B contains one of the main results of the book. It is the proof of the following theorem: Let S be a compact monoid which is not a group and the group of units is not open and there exists a neighborhood of 1 containing no other idempotent. Then there exists a one parameter semigroup which immediately leaves the group of units never to return (closure of its image is nontrivial and intersects the group of units only in the identity).

The basic ideas involved in the proof go back to those used in the solution of Hilbert's Fifth Problem created by Gleason, Iwasawa, Montgomery, and Zippin, as exposited in Montgomery and Zippin's book, together with the use of the approximation theorem for compact topological groups. The proof goes as follows: One first defines a local semigroup called nucleus and local one-parameter semigroups called rays. Then the authors show it suffices to prove the existence of a ray which immediately leaves the group of units, since the ray can be extended by the usual tricks. Thus the theorem is a local theorem.

Now by hypothesis we have a compact semigroup S with identity 1 and group of units H such that H is not open and is contained in a neighborhood U which is without idempotents different from 1. Now under these hypotheses it is not too difficult to show that there exists a net x on U so that $\mathrm{x}(i) \in U-H$ and $\mathrm{x}(i) \to 1$ and for each $x = \mathrm{x}(i)$ there exists an $n(i)$ such that $x^1, x^2, x^3, \cdots, x^{n(i)} \in U$, $x^{n(i)+1} \notin U$ since otherwise the monogenic semigroup generated by x lies in U so its idempotent must be 1, so it must be a group, so x is in H, a contradiction. Thus the powers of members of $U-H$ move down and away from H and U. Let Y equal $(S-U)^*$ so $Y \cap H = \emptyset$.

Now the main idea of the proof (which goes back to Gleason) enters. Intuitively the arc is given by $t \to x^t$, but it is not clear what x^t means. However, we can give a meaning to $x^{p/q}$ and take a limit by Tychonoff. To do this let $f(i)(p) = x^{[p/q]}$. Now taking a pointwise limit of a subsequence of the $f(i)$ (which exists by Tychonoff) we obtain a homomorphism f which may not be continuous but with $f(1) \in U \cap Y$. Thus f is a good algebraic map but might be horribly discontinuous. To measure the discontinuities of f (following Gleason again) the authors introduce the group C of elements of S such that there is a net P of $(0, 1/2)$ convergent to zero such that $f(P) = x$. They show that C is a compact connected commutative subgroup of the nucleus. Also, as usual, f is continuous iff $C=0=f(0)$. Now they consider the image of f in its nucleus (S, m). Then by some technical but fairly straightforward arguments they show that (S, m) can be cut down to an abelian nucleus whose set of units is a compact group where $S\times H$ is always defined and there is a homomorphism not necessarily continuous such that

$$(*)\qquad \begin{aligned} &\phi:(S, m) \twoheadrightarrow ([0, 1], +), \qquad \phi^{-1}(r) = sH \qquad \text{for some } s \quad \text{and} \\ &\qquad\qquad \phi(x) + \phi(y) \leqq 1 \quad \text{implies } xy \text{ is defined.} \end{aligned}$$

The idea is to consider $C \times (\text{image } f)$. This has a coordinate system $C\times [0, 1]$ which is fine algebraically but the second coordinate may be wildly discontinuous.

Now under the above situation they finally prove that there exists a ray r such that $\phi \circ r=\text{id}$ on $[0, 1]$. This is done as follows: Starting from (*) choose a subnucleus T of S so that if two members of T multiply in S they multiply in T and lie in T, ϕ carries T onto $[0, 1]$ and $T\cap tH=(T\cap H)t$ for all t in T. The class of such T's is inductive and thus T can be chosen minimal with respect to these properties. Thus no proper subnucleus of T has all these properties. Now the idea is to show ϕ is one-to-one on T. They now let $H=H\cap T$ and assume S has no proper subnucleus satisfying the above noted conditions. Now they pull out the big guns, namely the approximation theorem for compact groups by Lie groups and the Gleason local cross section theorems. Using these theorems they choose N to be a small closed normal subgroup so that H/N is a Lie group and consider the nucleus $(S/N, m)$. Then by (*) $\phi_N:(S/N, m_N)\rightarrow([0, 1], +)$ is defined. Then by the Gleason section theorem there is an arc K at N of S/N which is a cross section for the action of H/N on S/N. Then letting H_1 be a cell neighborhood of the identity which does not contain any nontrivial subgroups of the nucleus KH_1, a peripheral arc construction by Tychonoff approximation here must be continuous since C would be 1 and it is easy to see there are no idempotents in the neighborhood. Now let $T=\phi_N^{-1}(f_N[0, 1])$. One can check T satisfies (*). Thus $S=T$ and so $N=H$. Thus by the approximation theorem for locally compact topological groups, $H=1$, $N=1$, and f_N is a ray and $\phi_N=\phi$ is 1:1 since $H=1$. Thus ends the proof of the theorem first proved by Mostert and Shields. A discussion of the local cross-section theorem with some proofs is provided in Appendix II.

The authors (modulo A. Borel's deep results on the universal and classifying spaces of compact Lie groups acting on locally compact spaces and the cohomology rings of the relevant spaces) state and prove in an appendix the following theorem of A. Borel and the authors which generalizes earlier more elementary results of Conner. Let S be a compact connected semigroup with identity and U a compact connected abelian group of automorphisms of S. Then the set of fixed points of U on S is a compact connected subsemigroup which meets the minimal ideal. This result is immediately used to show the following important corollary. Let S be a compact connected semigroup with identity and A be a compact connected abelian group of units. Then the centralizer of A is a compact connected subsemigroup containing A and meeting the minimal ideal.

Now using all of the above results the authors prove Koch's result (modulo the result of Mostert aud Shields) that any compact connected monoid contains an *abelian* compact connected subsemigroup which contains 1 and meets the minimal ideal $M(S)$. Their proof goes as follows: first push the groups to one point in $M(S)$ so $M(S)$ becomes acyclic. It suffices to prove the theorem here. Next let U be an entourage of the uniform structure of S and let T_U be maximal in the collection of all compact abelian subsemigroups containing 1 with $M(T)$ connected and T_U U-connected. They wish to show $T_U \cap M(s) \neq \emptyset$. Suppose not. Then $M(T_U)$ is an abelian group with identity e which acts on the acyclic space eSe by $s \rightarrow g^{-1}sg$ and leaves e and some points of the acyclic space $M(eSe)$ fixed. By the theorems proved before the fixed point set is connected and the centralizer Z of $U(T_U)$ is connected, contains e, and meets $U(S)$. Now if $U(e) \cap Z$ contains another idempotent f, then $T_U \cup fM(T_U)$ is larger and violates the maximal definition of T_U. If $U(e) \cap Z$ has no idempotents except e then there exists a ray in $eSe \cap Z$ starting from e and leaving $U(e)$ with image in I. Then $T_U \cup M(T_U)I$ violates the maximal definition of T_U. Thus T_U must meet $M(S)$. Now limiting over all U yields a compact connected abelian semigroup T such that T hits 1 and $M(S)$. Done.

Let T be a compact connected monoid. Then the authors say T is *irreducible* iff T contains no proper compact connected subsemigroups meeting $M(T)$, and containing 1. Zorn implies that for each compact connected monoid S there exists an irreducible subsemigroup T containing 1 and meeting $M(S)$.

The remainder of Chapter B is mainly devoted to determining the properties of irreducible monoids. First from Koch's Theorem *irreducible semigroups are abelian.* Then they are able to prove with some ease that if T is irreducible then the group of units of T is trivial and T/H (H is the Green relation tHt' iff $Tt = Tt'$ and $tT = t'T$) is a connected totally ordered semigroup with 0 and 1 as endpoints (i.e., a thread or I-semigroup). Finally the authors define a *hormos* which is