CAUCHY PROBLEMS INVOLVING A SMALL PARAMETER

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The purpose of this note is to indicate how certain asymptotic methods developed for ordinary differential equations can be extended and applied to initial-boundary value problems for nonlinear parabolic and hyperbolic equations. This is done by considering the initial-boundary value problem as a Cauchy problem for an ordinary differential equation in an abstract space.

We consider the initial value problem

$$(1) \qquad \epsilon(dv/dt) - A(t, \epsilon)v = f(t, v, \epsilon), \qquad 0 \leqq t \leqq T, \quad v(0) = \mathring{v}(\epsilon)$$

where v is an element of a Banach space E and $\epsilon > 0$ is a small parameter. The (possibly unbounded) linear operators A are assumed to have a common domain of definition $\mathfrak{D}$ independent of (t, ϵ), and the function f is assumed to have continuous derivatives with respect to t, ϵ and continuous Fréchet derivatives with respect to v. Finally, $\mathring{v}(\epsilon) \in \mathfrak{D}$ has continuous derivatives with respect to ϵ.

We will outline here a method for finding an expansion for the solution of (1) which is valid as $\epsilon \to 0$.

1. **Formal method.** We begin by formally describing the procedure. These steps will be justified by Theorems 1–3. Suppose

(I) the operator $A(t, \epsilon)$ has a bounded inverse for each (t, ϵ) and $A(t, \epsilon)A^{-1}(0, 0)$ has continuous derivatives with respect to (t, ϵ).

Assuming for the moment that (1) has a solution for $\epsilon > 0$, we differentiate (1) successively with respect to ϵ and set $\epsilon = 0$ in the results. This gives the system of equations

$$(2a) \qquad - A(t, 0)v_0 = f(t, v_0, 0)$$

$$(2b) \qquad -[A(t, 0) + f_r(t, v_0(t), 0)]v_r = R_r(t), \qquad r = 1, 2, \cdots,$$

for the coefficients v_r of the Taylor expansion of v about $\epsilon = 0$. Next, we make the change of variables $t = \epsilon\tau$ in (1):

$$(3) \qquad dV/d\tau - A(\epsilon\tau, \epsilon)V = f(\epsilon\tau, V, \epsilon), \qquad V(0) = \mathring{v}(\epsilon).$$

By differentiating this successively with respect to ϵ and setting $\epsilon = 0$ in the results, we get

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(4a) $$dV_0/d\tau - A(0, 0)V_0 = f(0, V_0, 0), \qquad V_0(0) = \mathring{v}(0)$$

(4b) $$dV_r/d\tau - [A(0, 0) + f_r(0, V_0, 0)]V_r = \rho_r(\tau), \qquad V_r(0) = \mathring{v}_r, \qquad r = 1, 2, \cdots,$$

for the coefficients V_r of the Taylor expansion of the solution of (3) about $\epsilon = 0$. In (4b) $\mathring{v}_r$ is the coefficient of ϵ^r in the Taylor expansion about $\epsilon = 0$ of $\mathring{v}(\epsilon)$. We observe that for each r, R_r in (2b) depends only on $t, v_0, \cdots, v_{r-1}$, and ρ_r in (4b) depends only on $\tau, V_0, \cdots, V_{r-1}$.

If problems (2), (4) can be solved successively for the v_r, V_r, we can form the (possibly divergent) expansions

(5a) $$\sum_{r=0}^{\infty} v_r(t)\epsilon^r,$$ (5b) $$\sum_{r=0}^{\infty} V_r(t/\epsilon)\epsilon^r.$$

From analogy with the ordinary differential equations case ($A(t, \epsilon)$ bounded operators) the solution of (1) is expected to be represented by (5a) for t away from zero and by (5b) for t near zero. To obtain an expansion for the solution of (1) valid uniformly for $0 \leqq t \leqq T$, we employ a matching device. We observe that the expansion

(6) $$\sum_{r=0}^{\infty} v_r(\epsilon\tau)\epsilon^r$$

formally satisfies the problem (3). Expanding each $v_r(t) = \sum_{q=0}^{\infty} v_{r,q}t^q$ in its formal Taylor expansion and substituting these into (6) gives

(7) $$\sum_{r=0}^{\infty} U_r(\tau)\epsilon^r, \qquad U_r(\tau) = \sum_{q=0}^{r} v_{r-q,q}\tau^q, \qquad r = 1, 2, \cdots.$$

It is shown below that expansion (7) is like expansion (5a) for t near zero and like (5b) for t away from zero. We thus arrive at the expansion

(8) $$\sum_{r=0}^{\infty} [v_r(t) + V_r(t/\epsilon) - U_r(t/\epsilon)]\epsilon^r.$$

Expansion (8) is considered in three different cases:

(i) Abstract Parabolic Case where for each $(t, \epsilon) \in [0, T] \times [0, \epsilon_0]$, $-A(t, \epsilon)$ is the infinitesimal generator of an analytic semigroup of operators in E (see Kato [1]).

(ii) Abstract Hyperbolic Case where for each (t, ϵ), $-A(t, \epsilon)$ is the infinitesimal generator of a semigroup of class C_0 [2].

(iii) Parabolic Case where A is a positive definite elliptic operator in $E = L^\infty$.

2. **Assumptions.** We assume:

(II) The problem (2a) has an isolated solution which is infinitely differentiable for $0 \leqq t \leqq T$ and $f_v(t, v_0(t), 0) = 0$ for $0 \leqq t \leqq T$ (i.e., A accounts for the linearization of (1) about $v = v_0(t)$).

(III) The problem (4a) has a unique solution $V = V_0(\tau)$ which exists for $0 \leqq \tau < \infty$.

Finally, a crucial condition for our work is:

(IV) The resolvent set of $-A(t, \epsilon)$ includes the half plane $\{\mathrm{Re} Z \geqq -\delta\}$ for some $\delta > 0$.

3. **Results.** The proofs of the following theorems will be given elsewhere.

THEOREM 1 (ABSTRACT PARABOLIC CASE). *Let conditions* (I)–(IV) *and* (i) *be satisfied. Then for sufficiently small* $|\mathring{v}(0) - v_0(0)|_E$, *there exists a unique solution* $v = v(t, \epsilon)$ of (1) *for each small* ϵ. *Also, the problems* (2) *and* (4) *can be solved successively and*

$$v(t, \epsilon) \underset{E}{\sim} \sum_{r=0}^{\infty} [v_r(t) + V_r(t/\epsilon) - U_r(t/\epsilon)]\epsilon^r$$

where v_r, V_r, U_r *are determined from* (2), (4) *and* (7), *respectively.*

REMARKS. The notation $g(t, \epsilon) \underset{E}{\sim} \sum_{r=0}^{\infty} \alpha_r(t, \epsilon)\epsilon^r$ here means that for each $N = 1, 2, \cdots$, the function S_N defined by $\epsilon^{N+1} S_N = g(t, \epsilon) - \sum_{r=0}^{N} \alpha_r(t, \epsilon)\epsilon^r$ is bounded in the norm of E uniformly for $0 \leqq t \leqq T$, $0 < \epsilon \leqq \epsilon_0$.

The restriction on $|\mathring{v}(0) - v_0(0)|_E$ in Theorem 1 is primarily to ensure that $|V_0(\tau) - v_0(0)|_E \to 0$ as $\tau \to \infty$. Thus, its size depends on the nonlinearity f and the location of the spectrum of $-A(t, \epsilon)$.

THEOREM 2. (ABSTRACT HYPERBOLIC CASE). *Let conditions* (I)–(IV) *and* (ii) *be satisfied. Also, suppose* $|A(t, \epsilon) - A(s, \epsilon)|_E \leqq C|t - s|$ *for some constant* $C > 0$ (*independent of* ϵ) *and all* $0 \leqq t$, $s \leqq T$. *Then the conclusion of Theorem* 1 *remains valid.*

REMARKS. An important restriction imposed by the continuity condition on A in Theorem 2 is that the operator defined by the difference is a bounded operator. This condition does not appear in the parabolic case because of certain properties of analytic semigroups.

The proof of Theorems 1 and 2 rests on obtaining an estimate for the fundamental solution of problem (1). In particular, we have

LEMMA. *Let either the hypotheses of Theorem* 1 *or* 2 *be satisfied. Then* *for each* $\epsilon>0$, *there is a fundamental solution*, $U(t, s, \epsilon)$, *for the linear part of problem* (1) (*i.e.* (1) *with* $f=0$). *Moreover there are positive constants* K, η *such that*

$$|U(t,s,\epsilon)|_E \leqq K\exp[-\eta(t-s)/\epsilon] \qquad \text{for } 0 \leqq s \leqq t \leqq T, \quad 0<\epsilon \leqq \epsilon_0.$$

The proof of the following theorem rests on obtaining a similar estimate for the fundamental solution when $E=L^\infty$. Let Ω be a bounded domain in Euclidean n-space E^n with boundary $\partial\Omega$ and closure $\bar{\Omega}$. A point $x\in E^n$ is given by $x=(x_1, \cdots, x_n)$, and we use the notation $D_i=\partial/\partial x_i$. We shall denote by $\mathfrak{A}(x, t, \epsilon, D)$ a second order linear differential operator in L^∞ with real coefficients:

$$\mathfrak{A}(x, t, \epsilon, D) = \sum_{i,j=1}^{n} a_{ij}(x, t, \epsilon)D_iD_j + \sum_{i=1}^{n} a_i(x, t, \epsilon)D_i + a(x, t, \epsilon).$$

The coefficients of $\mathfrak{A}$ have continuous derivatives of all orders with respect to $(x, t, \epsilon)\in\bar{\Omega}\times[0, T]\times[0, \epsilon_0]$. Also, the matrix (a_{ij}) is symmetric and positive definite uniformly in (x, t, ϵ) (in particular at $\epsilon=0$). Finally, Ω is of class C^2.

Consider the initial-boundary value problem

$$\begin{aligned} &\epsilon u_t - \mathfrak{A}u = f(x, t, u, \epsilon), \qquad u = 0 \text{ on } \partial\Omega\times[0, T],\\ &u(x, 0, \epsilon) = \mathring{u}(x, \epsilon) \text{ on } \bar{\Omega}. \end{aligned} \tag{9}$$

Here u_t denotes $(\partial u/\partial t)$, and we assume f, $\mathring{u}$ have continuous derivatives of all orders. The formal considerations above proceed in the same way for this problem. We will denote by (2*), (4*), etc., those statements reinterpreted for (9). We then have

THEOREM 3 (PARABOLIC CASE). *Let* $\mathfrak{A}$, f, $\mathring{u}$ *be as above and let condition* (II*)–(IV*) *be satisfied. Then for* $|\mathring{u}(x, 0)-v_0(x, 0)|_{L^\infty}$ *sufficiently small, there is a unique solution of* (9) *for each small* ϵ. *Moreover*, (2*), (4*) *can be solved successively and*

$$u(x, t, \epsilon) \underset{L^\infty}{\sim} \sum_{r=0}^{\infty} [v_r^*(x, t) + V_r^*(x, t/\epsilon) - U_r^*(x, t/\epsilon)]\epsilon^r.$$

REMARKS. In (9), $\mathfrak{A}$ can be replaced by an elliptic operator of order $2m$ for any integer $m>0$ with a corresponding change in the boundary conditions (see e.g., Agmon [3]). The estimate in L^∞ for the fundamental solution of problem (9) is obtained from the maximum principle for parabolic equations.

In [4] Keller formally obtained an expansion for the solution of (9) in the linear case (i.e., f independent of v). His expansions involved the eigenfunctions of $\mathfrak{A}$. Since the Green's function for the linear part of (9) can be expanded in terms of the eigenfunctions of $\mathfrak{A}$, the expansion of Theorem 3 can be given in terms of these eigenfunctions. In the linear case the result agrees with that in [**4**].

The method outlined here is essentially the one developed by Vasil'eva [**5**] for ordinary differential equations. Her work suggests that these methods can be extended to treat systems of the form $u_t = g(t, u, v, \epsilon)$, $\epsilon v_t = A(t, \epsilon)v + f(t, u, v, \epsilon)$. Such extensions are presently being investigated. Finally, this work has been applied to problems involving the heat equation with nonlinear source in a domain with a slowly moving boundary.

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