MORSE THEORY ON BANACH MANIFOLDS

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S. Smale has conjectured, in an unpublished paper, that the Morse Theory on Hilbert mainfolds due to Palais and Smale [1], [4] can be extended to Banach manifolds. Under a different definition of nondegeneracy of critical points we have been able to make this extension. The result also extends Morse theory on Hilbert manifolds to a wider class of functions. I wish to thank R. Palais for several helpful suggestions.

Let f be a real-valued C^1 function on a C^1 Banach manifold X. A critical point x of f is said to be weakly nondegenerate if there exists a neighborhood U of x and a hyperbolic linear isomorphism L_x: $T_x(X)\to T_x(X)$ such that in the coordinate system of U, $df_{x+v}(L_x v)>0$ for all $x+v$ in U, $v\neq 0$. Then $T_x(X)$ splits into the direct sum of two invariant subspaces of L_x, $T_x(X)\cong T_x(X)_+\oplus T_x(X)_-$ such that the spectrum of L_x on $T_x(X)_+$ lies in the right half plane and the spectrum of L_x on $T_x(X)_-$ lies in the left half plane. The index of f at x is defined to be dim $T_x(X)_-$, and this term is well defined. A nondegenerate critical point of a function on a Hilbert manifold is weakly nondegenerate.

THEOREM 1. *Let f be a C^2 function on a C^2 paracompact manifold X without boundary modeled on a separable Banach space B. We assume that B has C^2 partitions of unity and a metric which is C^2 away from* 0. *If, in addition,*

(a) *f satisfies condition* (C) *of Palais and Smale with respect to a complete Finsler metric on X, and*

(b) *$q>q'$ are not critical values, and all the critical points in $f^{-1}((q,q'))$ are weakly nondegenerate of finite index,*

then there exists a homeomorphism θ: $f^{-1}[q,-\infty)\cong f^{-1}[q',-\infty)\cup h_i$ where a handle h_i of index q_i is added for each one of the finite number of critical points $x_i\in f^{-1}((q,q'))$ of index q_i.

REMARK. In the case of an infinite index, a similar result holds, provided that

$$df_{x+v}(L_x v) > \alpha(\|v\|_B) \qquad \text{for} \qquad 0\neq v\in T_x(M)_-\cap U$$

where α is a continuous function from $R^+\to R^+$.

THEOREM 2. *Let η be a vector bundle over a finite dimensional manifold N. Let the integral J: $L_k^p(\eta)_0 \to R$ be given by*

$$J(s) = \int_N (1 + | A(s) |^2)^{p/2} + B(s)du \qquad (p \geqq 2)$$

where A is a nonlinear (over-determined) elliptic system of order and weight k, $pk > \dim N$, and B is of order $k-1$ and weight pk.[1] *Then J is $C^{[p]}$ (C^∞ for p even) on the Sobolev space $L_k^p(\eta)_0$, and if the critical point v has the properties*:

(a) $v \in C^{k+\alpha}(\eta)$ *for any* $\alpha > 0$,

(b) *the bilinear form $d^2J_v(\ ,\ )$ extends to a nondegenerate form on $H_k(\eta)_0$,*

then v is a weakly nondegenerate critical point of J with finite index.

REFERENCES

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2. ———, *Ljusternik-Schnirelmann theory on Banach manifolds*, Topology, **5** (1966), 115–132.

3. ———, *Foundations of non-linear analysis*, Benjamin, New York, 1968.

4. S. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. (2) **80** (1964), 382–396. MR **29** #2820.

5. ———, *Morse theory on Finsler manifolds* (unpublished article).

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[1] See Chapter 16 of [3].