ON AXIOMS FOR B^*-ALGEBRAS

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Let B be a complex Banach algebra with an involution $x \to x^*$. Let H denote the set of selfadjoint (s.a.) elements of B and W the subset of H consisting of all $h \in H$ whose spectrum is entirely real. As in [3] we denote the spectral radius of $x \in B$ by $\nu(x)$. We prove the following result.

THEOREM. *Suppose that there exists $c > 0$ where $\nu(h) \geqq c\|h\|$ for all $h \in H$. Then W is closed in B.*

This theorem has consequences for the theory of B^*-algebras. Shirali and Ford [4] have recently shown that B is symmetric if $W = H$. Combining this and Lemma 2.6 of [6] with our theorem, we obtain the following result.

COROLLARY 1. *B is a B^*-algebra in an equivalent norm if and only if W is dense in H and, for some $c > 0$, $\nu(h) \geqq c\|h\|$ for all $h \in H$.*

As usual $x \in B$ is said to be normal if $xx^* = x^*x$. Let N denote the set of normal elements of B. Berkson [1] and Glickfeld [2] have shown (in case B has an identity) that B is a B^*-algebra in the given norm if $\|x^*x\| = \|x^*\|\|x\|$ for all $x \in N$. We obtain an analogous result for equivalence to a B^*-algebra.

COROLLARY 2. *B is a B^*-algebra in an equivalent norm if and only if, for some $c > 0$, the set of $x \in N$ for which $\|x^*x\| \geqq c\|x^*\|\|x\|$ is dense in N and contains H.*

We turn to the proof of our theorem. Let B_1 be the algebra obtained by adjoining an identity 1 to B and defining, as usual, $\|\lambda + x\| = |\lambda| + \|x\|$ and $(\lambda + x)^* = \bar{\lambda} + x^*$ where λ is complex and $x \in B$. We show that there exists $b > 0$ such that $\nu(y) \geqq b\|y\|$ for all y s.a. in B_1. For suppose otherwise. Then there exists a sequence $\{\lambda_n + h_n\}$, with λ_n real and $h_n \in H$, such that $|\lambda_n| + \|h_n\| = 1$ and $\nu(\lambda_n + h_n) \to 0$. By

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taking a subsequence if necessary we may suppose that $\lambda_n \to \lambda$ for some λ real. Now

$$\mathrm{sp}(\lambda + h_n) = (\lambda - \lambda_n) + \mathrm{sp}(\lambda_n + h_n).$$

If $\lambda \neq 0$, this relation and $\nu(\lambda_n + h_n) \to 0$ shows that $0 \notin \mathrm{sp}(h_n)$ for n sufficiently large. For these n, h_n^{-1} exists in B_1, which is impossible. Since $\lambda = 0$, we have $\|h_n\| \to 1$ whereas $\nu(h_n) \to 0$, contrary to our hypotheses.

Thus there is no loss of generality in assuming that B has an identity 1. Now we take $h \in W$. Let A_0 be the *-subalgebra of B generated by 1 and h and let A be its closure in B. Since the involution in B is continuous by Theorem 3.4 of [5], we see that A is a commutative closed *-subalgebra of B. For each s.a. element $x \in A_0$, $\mathrm{sp}(x|B)$ is real and, by Theorem 1.6.11 of [3], $\mathrm{sp}(x|B) = \mathrm{sp}(x|A)$. Since each s.a. element u of A is the limit of s.a. elements of A_0, Gelfand theory applied to A shows that $\mathrm{sp}(u|A) = \mathrm{sp}(u|B)$ is real. Let $\mathfrak{M}$ be the space of maximal ideals of A. For each $M \in \mathfrak{M}$, $|\exp(ih)(M)| = 1$. Write $\exp(ih) = s + it$ where s and t are s.a. in A. Then, since $s(M)$ and $t(M)$ are real-valued, we obtain

$$\begin{aligned} 1 &= \nu(\exp(ih)) \geqq \max(\nu(s), \nu(t)) \geqq c \max(\|s\|, \|t\|) \\ &\geqq (c/2)\|s + it\| = (c/2)\|\exp(ih)\|. \end{aligned}$$

Therefore

$$\|\exp(ih)\| \leqq 2c^{-1}, \qquad h \in W. \tag{1}$$

Inasmuch as the mapping $x \to \exp(x)$ is continuous on B, the relation (1) persists on the closure of W in B which, by the continuity of the involution, lies in H. Now take any w in that closure. Let B_0 be a maximal commutative *-subalgebra of B containing w and let $\mathfrak{M}$ be its space of maximal ideals. Suppose that $a + bi \in \mathrm{sp}(w|B) = \mathrm{sp}(w|B_0)$ (see Theorem 4.1.3 of [3]) where a, b are real. As $w = w^*$, then also $a - bi \in \mathrm{sp}(w)$. Choose M_1, M_2 in $\mathfrak{M}$ such that $w(M_1) = a + bi$ and $w(M_2) = a - bi$. Then $|\exp(iw)(M_1)| = \exp(-b)$ and $|\exp(iw)(M_2)| = \exp(b)$. For any $t > 0$, $tw \in \overline{W}$. Therefore we obtain

$$2c^{-1} \geqq \nu(\exp(itw)) \geqq \max(\exp(-tb), \exp(tb)).$$

It follows that $b = 0$ so that $\mathrm{sp}(w)$ is real.

It may be noted that an example of Kakutani given on p. 282 of [3] shows that one can have a sequence of elements in a Banach algebra B with purely real spectrum approaching an element whose spectrum is not entirely real.

We now turn to Corollary 2. A standard argument (see p. 191 of

[3]) shows that $\nu(h) \geqq c\|h\|$ for all $h \in H$. Then the involution on B is continuous by Theorem 3.4 of [5] so that $\|x^*x\| \geqq c\|x^*\|\|x\|$ for all $x \in N$. By Theorem 4.2.2 of [3], $W = H$ and we may apply Corollary 1.

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