ELEMENTARY CYCLES OF FLOWS ON MANIFOLDS

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There are two very natural notions of equivalence of flows (see [1], [2]) on a manifold. One is the existence of a homeomorphism mapping orbits onto orbits, preserving the natural orientation of orbits but not necessarily their natural parametrisation. The second requires that the homeomorphism alter natural parametrisations by at most a positive constant multiple. We call the first relation on flows *orbit-equivalence* and the second *flow-equivalence*. There are obvious localisations of these relations. In general flow-equivalence is strictly stronger than orbit-equivalence. However, it is a consequence of the theorem of Hartman [5], [6], [7] and Grobman [3], [4] that the local notions of equivalence are the same at elementary (see [1], [2]) rest-points. The purpose of this note is to announce a similar result for elementary cycles. M. Shub has informed me that he and C. Pugh have also obtained this result.

1. **Preliminaries.** Let $\phi: \mathbf{R}\times X\to X$ be a C^1 flow on a C^∞ manifold X. We write $\phi_x(t)=\phi^t(x)=\phi(t, x)$, so that, for fixed $x\in X$, $\phi_x: \mathbf{R}\to X$ is C^1 and, for fixed $t\in \mathbf{R}$, $\phi^t: X\to X$ is a C^1 diffeomorphism. Let U be open in X. For fixed $x\in U$ let I_x denote the component of $(\phi_x)^{-1}(U)$ containing 0, and let D_U denote $\bigcup_{x\in U} I_x\times\{x\}$.

Now suppose that Ψ is a C^1 flow on a C^∞ manifold Y and that A and B are subsets of X and Y respectively. We say that A is *flow-equivalent* to B (with respect to the given flows) if there exist open neighbourhoods U of A and V of B and a homeomorphism $h: U\to V$ such that $h(A)=B$ and, for all $(t, x)\in D_U$,

$$h\phi(t, x) = \Psi(\alpha(t), h(x)),$$

where $\alpha: \mathbf{R}\to\mathbf{R}$ is a multiplication by some positive constant. In this case h maps orbit components of ϕ in U onto orbit components of Ψ in V, preserving orientation.

Let $\nu\in GL(\mathbf{E})$, where $\mathbf{E}$ is a finite dimensional real normed linear space. Let $\mathbf{F}$ be the largest invariant subspace of $\mathbf{E}$ on which ν has no complex eigenvalues of modulus 1. We call $\nu|\mathbf{F}$ the *hyperbolic part* of ν.

Recall [8] that we may associate with any hyperbolic linear automorphism of a k-dimensional real normed linear space its *suspension*, which is a flow on a $(k+1)$-dimensional manifold. It has precisely one cycle, corresponding to the unique periodic point 0 of the automorphism.

2. **Results.** The following is an analogue, for cycles, to the theorem of Hartman and Grobman for rest-points:

THEOREM 1. *Let ϕ be a C^1 flow on X, and let C be an elementary cycle of ϕ. For any $x \in C$, let D be the unique cycle of the suspension of the hyperbolic part of $T_x\phi^\tau$, where τ is the period of C. Then C is flow-equivalent to D.*

As corollaries one deduces

THEOREM 2. *Let C and D be elementary cycles of C^1 flows. Then C is flow-equivalent to D if and only if $m(C)=m(D)$, $n(C)=n(D)$, and $m^-(C)-m^-(D)$ and $n^-(C)-n^-(D)$ are even, where m, n, m^- and n^- here denote the numbers of expanding, contracting, real negative expanding and real negative contracting characteristic multipliers.*

and

THEOREM 3. *Elementary cycles of C^1 flows are flow-equivalent if and only if they are orbit-equivalent.*

The proof of Theorem 1 reduces, in effect, to a proof that there exists, at x, a local cross-section that is invariant under ϕ^τ. By taking a suitable chart and using a bump-function we may reduce this to the following

LEMMA 4. *Let $\nu: \boldsymbol{E} \to \boldsymbol{E}$ be a hyperbolic linear automorphism, let $f: \boldsymbol{E} \to \boldsymbol{E}$ be a homeomorphism and let $\zeta: \boldsymbol{E} \to R$ be a linear map. Suppose that, for some $d \geqq 0$, $f-\nu$ and ζ vanish at 0 and on $\{x \in \boldsymbol{E}; \|x\| \geqq d\}$ and are Lipschitz, the former with constant κ. Then, if κ is sufficiently small, there exists a continuous map $\theta: \boldsymbol{E} \to R$ such that $\theta = \theta f + \zeta$.*

Subject to the condition $\theta(0)=0$, the map θ is uniquely defined on the stable and unstable manifolds of the origin with respect to f. Elsewhere, however, it is not unique. In the proof [**9**] of the lemma an explicit map θ is constructed.

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