LOCALIZED SOLUTIONS OF NONLINEAR WAVE EQUATIONS

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We consider complex valued solutions ϕ of nonlinear wave equations of the form

$$\Box\phi = \phi_{tt} - \Delta\phi = -\phi v(|\phi|^2) \tag{1}$$

where v is the derivative of a positive definite potential V. That is

$$\frac{dV(a)}{da} = v(a) \quad \text{and} \quad V(a) \geqq 0 \qquad \text{with } V(a) = 0 \quad \text{iff } a = 0. \tag{2}$$

We suppose $v(0) = m^2 > 0$.

A solution ϕ with finite energy is called localized if there is an $\epsilon > 0$ such that

$$\sup_x |\phi(x, t)| = M(t) > \epsilon \tag{3}$$

whenever ϕ exists.

THEOREM. *If, for some* a_0

$$V(a_0) < m^2 a_0 \tag{4}$$

then equation (1) *has localized solutions.*

The proof is based on the conservation of *energy* $\mathcal{E}$ and *charge* $\mathcal{Q}$

$$\mathcal{E} = \int \left\{ |\phi_t|^2 + \sum_{i=1}^{N} |\phi_{x_i}|^2 + V(|\phi|^2) \right\} dx, \tag{5}$$

$$\mathcal{Q} = \operatorname{Im} \int (\phi_t \bar{\phi}) dx. \tag{6}$$

Suppose that $|\phi|^2 < \epsilon$. Then, from (2)

$$V(|\phi|^2) > (m^2 - \delta)|\phi|^2 \tag{7}$$

where δ tends to zero if ϵ does.

By the Schwartz inequality and (7) we easily deduce that

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$$\mathcal{Q} \leqq \frac{1}{2\sqrt{m^2 - \delta}} \mathcal{E}. \tag{8}$$

However, consider now the initial data

$$\phi(x, 0) = a_0\eta(x), \qquad \phi_t(x, 0) = ima_0\eta(x), \tag{9}$$

where

$$\begin{aligned} \eta(x) &= 1 \qquad \text{for } |x| < R, \\ &= R + 1 - |x| \qquad \text{for } R \leqq x \leqq R + 1, \\ &= 0 \qquad \text{for } R + 1 \leqq |x|. \end{aligned}$$

It is easy to see that if (4) holds, then for large enough R

$$\mathcal{Q} > \frac{1}{2\sqrt{m^2 - \delta}} \mathcal{E}. \tag{10}$$

Since $\mathcal{Q}$ and $\mathcal{E}$ are independent of time, it then follows that $M(t) > \epsilon > 0$ whenever ϕ exists. This shows that ϕ with initial data (9) is localized.

Next, we point out that for some equations of the type (1), there is a particularly interesting family of localized solutions.

Consider a function of the form

$$\phi_{\nu,\alpha,a}(x, t) = e^{i\nu(\alpha t - \beta x)}\eta(\alpha x - \beta t + a). \tag{11}$$

This is a solution of equation (1) if $\alpha^2 - \beta^2 = 1$, $\nu^2 < m^2$ and if η satisfies

$$\Delta\eta = \eta v(|\eta|^2) - \nu^2\eta \tag{12}$$

with η vanishing exponentially as $|x| \to \infty$.

I have shown, by numerical integration of equations (5), (6), and (12), that there exist potentials V (in one-, two-, and three-dimensions) for which solutions ϕ of the form (11) exist and satisfy (10).

Consider now the initial conditions

$$\Phi = \sum_i \phi_{\nu_i\alpha_i a_i}(x, t_0), \qquad \Phi_t = \sum_i \frac{d}{dt} \phi_{\nu_i\alpha_i a_i}(x, t_0). \tag{13}$$

If each $\phi_{\nu_i\alpha_i a_i}(x, t)$ satisfies (10), then it is easy to see that if each $|a_i - a_j| = R_{ij}$ is sufficiently large, the initial data (13) also satisfy (10), and so the solution $\Phi(x, t)$ with these initial data is localized.

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