DOMAINS BOUNDED BY ANALYTIC JORDAN CURVES

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It is well known that repeated application of the Riemann mapping theorem shows that any finitely connected plane domain is conformally equivalent to a domain whose boundary components are analytic curves. A corresponding result has not previously been shown for arbitrary domains of infinite connectivity—the only cases being those domains for which the Koebe conjecture is known to be true (cf. [2]).

We will show that *any domain with countably many boundary components is conformally equivalent to a domain bounded by analytic Jordan curves and points.*

1. A *limit boundary component* of a domain is a boundary component, at least one of whose points is a point of accumulation of points on other boundary components. A *weak* limit boundary component of a domain is a point limit boundary component which corresponds to a point under *every* conformal map of the domain. A *circle domain* is a domain bounded by circles and points.

In a recent paper [2, Theorem 4] the author has obtained a new proof, using quasiconformal mappings, of the following result due to Strebel: *If D is a domain with weak limit boundary components, then D is conformally equivalent to a circle domain.*

An examination of this proof shows that the assumption that the limit boundary components are point boundary components and that they are weak is used *only* in the last step—to show that the images of the *limit* boundary components are points. Consequently, we obtain

THEOREM 1. *Let D be an arbitrary planar domain. Then D is conformally equivalent to a domain K, all of whose isolated boundary components are circles or points.*

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REMARK. For the proof of our main theorem which follows, a weaker version of the above would suffice—the conclusion being that all isolated boundary components are analytic Jordan curves or points. A proof of this, which is more elementary than the proof indicated above of Theorem 1, will appear in [3].

2. Let $\Gamma^0 = \Gamma^0(D)$ denote the collection of boundary components of the plane domain D. We form the sequence $\Gamma^\alpha = \Gamma^\alpha(D)$ (cf. [1], also [4])

$$\Gamma^0 \supseteq \Gamma^1 \supseteq \cdots \supseteq \Gamma^\omega \supseteq \Gamma^{\omega+1} \supseteq \cdots \supseteq \Gamma^\alpha \supseteq \cdots$$

where ω denotes the ordinal type of the natural numbers and α is an arbitrary ordinal number. If α has a predecessor $\alpha-1$, then Γ^α is defined as the subset of $\Gamma^{\alpha-1}$ of components which contain points which are limit points of the components in $\Gamma^{\alpha-1}$ so that if Γ^0 is given a metric space structure (see [1]) then Γ^α is the *derived* set of $\Gamma^{\alpha-1}$. If α is a limit ordinal, we define $\Gamma^\alpha = \bigcap_{\beta<\alpha} \Gamma^\beta$.

We recall [1] that if Γ^0 is countable then there exists a (first) ordinal η such that $\Gamma^{\eta+1}$ is empty. If Γ^0 is uncountable, then the sequence Γ^β does not terminate. The intersection $\bigcap \Gamma^\beta$ over all β is called the *perfect kernel* of Γ^0. Every component in the perfect kernel is a limit boundary component of other components in the perfect kernel. Note that the "ordinal type" of a boundary component is invariant under conformal maps of the domain.

3. Our main result is

THEOREM 2. *Let D be a domain with countably many boundary components. Then D is conformally equivalent to a domain bounded by analytic Jordan curves and points.*

A brief outline of the proof, by transfinite induction on α, follows. The complete proof will appear in [3].

Consider the following statement: (We use the term "analytic Jordan curves" to include points.)

$S(\alpha)$: There exist conformal maps $f(\gamma): D \to D_\gamma$ for all $\gamma \leqq \alpha$, and conformal maps $h(\gamma, \delta): D_\gamma \to D_\delta$ for $\gamma \leqq \delta \leqq \alpha$ such that (for $\gamma \leqq \delta \leqq \epsilon \leqq \alpha$):

(i) $h(\delta, \epsilon) \circ h(\gamma, \delta) = h(\gamma, \epsilon)$.

(ii) $h(\gamma, \delta)$ is the restriction of a conformal map of the domain bounded by the elements of $\Gamma^\gamma(D_\gamma)$.

(iii) $f(\delta) = h(\gamma, \delta) \circ f(\gamma)$.

(iv) The elements of $\Lambda^\alpha(D_\alpha) = \Gamma^0(D_\alpha) - \Gamma^\alpha(D_\alpha)$ are analytic Jordan curves.

The crucial condition is (iv). For $\alpha = \eta + 1$ this says (since $\Gamma^{\eta+1} = 0$)

that the domain $D_{\eta+1}$, which is conformally equivalent to D, is bounded by analytic Jordan curves.

$S(0)$ is trivially true for $f(0)=h(0, 0)=$identity. Suppose $S(\beta)$ is true for all $\beta<\alpha$. We show the truth of $S(\alpha)$.

Case 1. If α is not a limit ordinal and hence has a predecessor $\alpha-1$, then there exists a conformal map $f(\alpha-1)$: $D\to D_{\alpha-1}$ such that $\Lambda^{\alpha-1}(D_{\alpha-1})$ consists of analytic Jordan curves. Let $h(\gamma, \alpha)$ be the restriction to $D_{\alpha-1}$ of the map obtained by application of Theorem 1 to the domain bounded by the elements of $\Gamma^{\alpha-1}(D_{\alpha-1})$. The maps $f(\alpha)$ and $h(\gamma, \alpha)$ for arbitrary γ are defined so that conditions (i) and (iii) hold. Using the fact that an analytic Jordan curve, lying in the domain of analyticity of a conformal map, is carried by the map into an analytic Jordan curve, conditions (i)–(iv) for α can be immediately verified.

Case 2. If α is a limit ordinal it is necessary to use a diagonalization procedure and a normal family argument (see [**3**]).

From the proof of Theorem 2 we obtain the following generalization:

COROLLARY 1. *An arbitrary domain (countably or uncountably many boundary components) is conformally equivalent to a domain, all of whose boundary components except those in the perfect kernel are analytic Jordan curves.*

We obtain also

COROLLARY 2. *If every boundary component of an arbitrary domain D is a nondegenerate continua containing at least one point which is not a point of accumulation of points on other boundary components, then D is conformally equivalent to a domain bounded entirely by nondegenerate analytic Jordan curves. (The hypothesis implies that D has at most countably many boundary components.)*

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