ON THE RELATIONS BETWEEN TAUT, TIGHT AND HYPERBOLIC MANIFOLDS

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Communicated by Richard Palais, May 22, 1969

In [1], Professor Kobayashi defined hyperbolic and complete hyperbolic manifolds. In [2], Professor Wu defined tight and taut complex manifolds. The purpose of this paper is to show that these concepts are related in the following way:

complete hyperbolic $\Rightarrow$ taut

taut $\underset{\not\Leftarrow}{\Rightarrow}$ hyperbolic

hyperbolic $\Leftrightarrow$ tight (with respect to some metric)

It seems likely that taut implies complete hyperbolic, but I cannot prove that at the present time. Don Eisenman has obtained these results concurrently by a slightly different method.

We begin by recalling the definition of the Kobayashi pseudo-distance d_M associated to the complex manifold M. Let p and q be points in M. By a *chain* α from p to q, we mean a sequence $p=p_0, p_1, \cdots, p_k=q$ of points in M, points $a_1, \cdots, a_k$ in the unit disk $D=\{z\in C \mid |z|<1\}$ and holomorphic maps $f_1, \cdots, f_k$ of D into M with $f_i(0)=p_{i-1}$ and $f_i(a_i)=p_i$. The length $|\alpha|$ of α is defined by

$$|\alpha| = \sum_{i=1}^{k} d(0, a_i) = \sum_{i=1}^{k} \log \frac{1+|a_i|}{1-|a_i|}$$

where d is the Poincaré-Bergman distance on D. It is given by the metric $ds^2=dzd\bar{z}/(1-|z|^2)^2$. We set $d_M(p, q)=\inf_{\alpha\in A}|\alpha|$, where A is the set of all chains from p to q. It is easy to see that d_M is a pseudo-distance on M. If d_M is an actual distance, we say that M is *hyperbolic. M* is called *complete hyperbolic* if d_M is a complete metric, i.e., if all Cauchy sequences converge. Kobayashi (see [1, §8]) has shown that complete hyperbolic implies that all bounded subsets have compact closure.

It follows immediately from the definition of d_M and d_N, that if $f: M\to N$ is holomorphic and $p, q\in M$, then $d_N(f(p), f(q))\leqq d_M(p, q)$. The classical Schwarz-Pick lemma implies that the Kobayashi distance d_D on the unit disk D is the same as the Poincaré-Bergman distance d.

Let $\mathfrak{A}(N, M)$ denote the set of holomorphic maps from N into M. A sequence $\{f_i\}$ in $\mathfrak{A}(N, M)$ is called *compactly divergent* if given any compact K in N and compact K' in M, there exists j such that $f_i(K) \cap K' = \emptyset$ for all $i \geqq j$. Fix a metric ρ on M which induces its topology. $\mathfrak{A}(N, M)$ is called *normal* if every sequence in $\mathfrak{A}(N, M)$ contain a subsequence which is either uniformly convergent on compact sets or compactly divergent. M is said to be *taut* if $\mathfrak{A}(N, M)$ is normal for every N. (M, ρ) is said to be *tight* if $\mathfrak{A}(N, M)$ is equicontinuous for every N. It should be noted that tautness is an intrinsic property of M, while tightness depends on the metric ρ.

For the rest of the paper, we shall use the following notation.

(1) p and q are distinct points of M.

(2) $B = \{(w_1, \cdots, w_n) \mid |w_1|^2 + \cdots + |w_n|^2 < 1\}$ is a coordinate neighborhood centered at p, such that $q \notin B$.

(3) $B_s = \{(w_1, \cdots, w_n) \mid |w_1|^2 + \cdots + |w_n|^2 \leqq s^2 < 1\} \subset B$.

(4) ρ is a metric on M which induces its topology.

(5) $V_s = \{p' \in M \mid \rho(p', p) < s\}$.

(6) $D_\delta = \{z \in D \mid |z| < \delta < 1\}$.

An ordered pair (r, δ) of strictly positive numbers is said to *satisfy property A* (relative to the choices above) if for every holomorphic map $f\colon D \to M$ with $f(0) \in B_r$, we have $f(D_\delta) \subset B$.

LEMMA. *Let M, p, q and B be as above. If there exists a pair (r, δ) satisfying property A, then $d_M(p, q) > 0$.*

PROOF. Choose a constant $c > 0$ such that $d_D(0, a) \geqq c d_{D_\delta}(0, a)$ for all $a \in D_{\delta/2}$.

Let $\alpha = \{p = p_0, p_1, \cdots, p_l = q; a_1, \cdots, a_l; f_1, \cdots, f_l\}$ be a chain from p to q. Without loss of generality, we can assume that $a_1, \cdots, a_k \in D_{\delta/2}$, $p_0, p_1, \cdots, p_k \in B_r$ and that $p_k \in \partial B_r$. Now

$$|\alpha| \geqq \sum_{i=1}^{k} d_D(0, a_i) \geqq c \sum_{i=1}^{k} d_{D_\delta}(0, a_i) \geqq c \sum_{i=1}^{k} d_B(p_{i-1}, p_i)$$

$$\geqq c d_B(0, p_K) = c' \quad \text{where } c' \text{ is constant} > 0.$$

Thus $d_M(p, q) \geqq c' > 0$. Q.E.D.

PROPOSITION 1. *If (M, ρ) is tight, then M is hyperbolic. If M is hyperbolic, then (M, d_M) is tight.*

PROOF. Assume (M, ρ) is tight. Using the notation above, there exists $\epsilon > 0$ such that $V_{2\epsilon} \subset B$. Since $\mathfrak{A}(D, M)$ is equicontinuous, there exists $\delta > 0$ such that if $f\colon D \to M$ is holomorphic with $f(0) \in V_\epsilon$, then $f(D_\delta) \subset V_{2\epsilon} \subset B$. Choose $r > 0$ such that $B_r \subset V_\epsilon$. Then (r, δ) satisfies property A. By the lemma, $d_M(p, q) > 0$. Since p and q were arbitrary distinct points, M is hyperbolic. The second statement is trivial. Q.E.D.

PROPOSITION 2. *If M is taut, then M is hyperbolic.*

PROOF. Assume M is not hyperbolic. Then there exist distinct points p and q with $d_M(p, q) = 0$. By the lemma, $(\frac{1}{2}, 1/n)$ does not satisfy property A for any n. Thus there exists a holomorphic map $f_n: D \to M$ with $f(0) \in B_{1/2}$ and $f_n(D_{1/n}) \not\subset B$. The sequence $\{f_i\}$ has no subsequence which is either uniformly convergent on compact sets or compactly divergent. Thus M is not taut. Q.E.D.

EXAMPLE. $D \times D - \{(0, 0)\}$ is hyperbolic, but it is neither taut nor complete hyperbolic.

PROPOSITION 3. *If M is complete hyperbolic, then M is taut.*

PROOF. Let N be another manifold. Since (M, d_M) is tight, $\mathcal{A}(N, M)$ is equicontinuous. Since M is complete hyperbolic, every bounded set in M is relatively compact. This implies that $\mathcal{A}(N, M)$ is normal (see Lemma 1.1 in [2]). Thus M is taut. Q.E.D.

Finally, we observe that if M is a hyperbolic Riemann surface, then M is complete hyperbolic. This follows from the fact that M is covered by D which is complete hyperbolic. By Proposition 5.5 of [1], M is complete hyperbolic.

REFERENCES

1. S. Kobayashi, *Invariant distances on complex manifolds and holomorphic maps*, J. Math. Soc. Japan **19** (1967), 460–480. MR **38** #736.

2. H. Wu, *Normal families of holomorphic mappings*, Acta Math. **119** (1967), 193–233. MR **37** #468.

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