COBORDISM OF REGULAR $O(n)$-MANIFOLDS

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A C^∞ manifold M together with a C^∞ action of $O(n)$ on M is said to be a regular $O(n)$-manifold if, for each $m \in M$, the isotropy group of m, $O(n)_m = \{g \in O(n) \mid gm = m\}$, is conjugate in $O(n)$ to $O(p)$ for some $p \leqq n$; $O(p)$ is understood to be imbedded in $O(n)$ in the standard way [3]. Compact regular $O(n)$-manifolds M_1^s, M_2^s are said to be (regularly) cobordant if there exists a compact regular $O(n)$-manifold W^{s+1} with ∂W^{s+1} equivariantly diffeomorphic to $M_1 \cup M_2$.

The set of cobordism classes of regular $O(n)$-manifolds of dimension s will be denoted by $\mathfrak{N}O(n)_s$. $\mathfrak{N}O(n)_*$ is a graded algebra over $\mathfrak{N}_*$, the cobordism ring of unoriented manifolds; addition is given by disjoint union, multiplication by cartesian product (with the diagonal action $g(m_1, m_2) = (gm_1, gm_2)$, $(m_1, m_2) \in M_1 \times M_2$) and $\mathfrak{N}_*$ acts by cartesian product (with the obvious action $g(m_1, m_2) = (m_1, gm_2)$, $(m_1, m_2) \in M_1 \times M_2$, $[M_1] \in \mathfrak{N}_*$, $[M_2] \in \mathfrak{N}O(n)_*$).

EXAMPLES. (A) Let M = point. Then $[M] \in \mathfrak{N}O(n)$. The submodule of $\mathfrak{N}O(n)$ (as a $\mathfrak{N}_*$ module) generated by $[M]$ [i.e. trivial $O(n)$ manifolds] is isomorphic to $\mathfrak{N}_*$ and we clearly have a decomposition $\mathfrak{N}O(n)_* = \mathfrak{N}_* \oplus \tilde{\mathfrak{N}}O(n)_*$.

(B) Any manifold with a differentiable involution is a regular $O(1)$ manifold.

(C) If M is a regular $O(n)$ manifold then by restricting the action to $O(n-1) \subset O(n)$ we get a regular $O(n-1)$ manifold. Since restriction respects cobordism there is an $\mathfrak{N}_*$ map $\rho\colon \mathfrak{N}O(n)_* \to \mathfrak{N}O(n-1)_*$.

(D) Given a regular $O(n)$ manifold M, one can extend the action to a regular $O(n+1)$ action on $O(n+1) \times_{O(n)} M$ and hence there is an $\mathfrak{N}_*$ map ext: $\mathfrak{N}O(n)_s \to \mathfrak{N}O(n+1)_{s+n}$.

(E) Let M be a regular $O(1)$ manifold and let P be an $O(n-1)$ principal bundle. Then $P \times M$ is an $O(n-1) \times O(1)$ manifold and $O(n) \times_{O(n-1) \times O(1)} P \times M$ is a regular $O(n)$ manifold. Hence, there is a homomorphism $h\colon \mathfrak{N}O(1) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(BO(n-1)) \to \mathfrak{N}O(n)_*$.

THEOREM. (i) *$\mathfrak{N}O(n)_*$ is a free $\mathfrak{N}_*$ module on countably many generators:*

(ii) *the algebra structure is given by $xy = 0$ for $x, y \in \tilde{\mathfrak{N}}O(n)_*$, $n > 1$,*

(iii) *$\rho \mid \tilde{\mathfrak{N}}O(n)_*$ is the zero map,*

(iv) *ext $\mid \tilde{\mathfrak{N}}O(n)_*$ is a monomorphism onto a direct summand of $\mathfrak{N}O(n+1)_*$; ext $\mid \mathfrak{N}_*$ is zero,*

(v) *h is an epimorphism.*

COROLLARY 1. *If M is a nontrivial regular $O(n)$ manifold without boundary then there exists a regular $O(n)$ manifold M', regularly cobordant to M, such that each isotropy group in M' is either conjugate to $O(1)$ or is trivial. In particular, $SO(n)$ acts freely on M'.*

COROLLARY 2. *If M and M' are regularly cobordant $O(n)$-manifolds such that each isotropy group in M and M' is conjugate to $O(1)$ or is trivial then there is a regular cobordism W between M and M' such that each isotropy group in W is conjugate to $O(2)$ or $O(1)$ or is trivial.*

Construction of generators. It is shown in [2] that $\mathfrak{N}O(1)_* = \sum_{k=2}^{\infty} \mathfrak{N}_*(BO(k))$. The isomorphism is constructed as follows: Let $E \rightarrow M$ be a differentiable k plane bundle over M and let $D(E)$, $S(E)$, $P(E)$ be respectively the unit disc bundle, sphere bundle, and projective bundle of E. Let $L \rightarrow P(E)$ be the disc bundle associated to the S^0 bundle $S(E) \rightarrow P(E)$. Then $L \cup_{S(E)} D(E)$ is the $O(1)$ manifold corresponding to $[E] \in \mathfrak{N}_*(BO(k))$. If ξ_r denotes the canonical line bundle over P^r associated to $S^r \rightarrow P^r$ then an $\mathfrak{N}_*$ basis for $\mathfrak{N}_*(BO(k))$ is given by the external products $\xi_{i_1} \times \xi_{i_2} \cdots \times \xi_{i_k}$ where $i_1 \geqq i_2 \cdots \geqq i_k \geqq 0$ [2]. Similarly, every principle $O(n-1)$ bundle is cobordant to a linear combination (with coefficients in $\mathfrak{N}_*$) of bundles $P(s_1 \cdots s_{n-1}) = S^{s_1} \times S^{s_2} \cdots \times S^{s_{n-1}} \times_{Q(n-1)} O(n-1)$ where $Q(n-1)$ is the product of $O(1)$ with itself $(n-1)$ times and $s_1 \geqq s_2 \cdots s_{n-1} \geqq 0$. Hence, by example E and (v) of the theorem we have

PROPOSITION 1. *The manifolds*

$$M(i_1, i_2 \cdots i_k; s_1 \cdots s_{n-1}) = h([\xi_{i_1} \times \xi_{i_2} \cdots \times \xi_{i_k}], [P(s_1 \cdots s_{n-1})])$$

with $i_1 \geqq i_2 \cdots \geqq i_k$; $s_1 \geqq s_2 \cdots \geqq s_{n-1}$ and $k \geqq 2$ generate $\mathfrak{N}O(n)_*$ as an $\mathfrak{N}_*$ module.

Note that the dimension of $M(i_1, \cdots i_k; s_1 \cdots s_{n-1})$ is

$$\sum_{j=1}^{k} (i_j + 1) + \sum_{j=1}^{n-1} s_i + \frac{n(n-1)}{2} \cdot$$

These generators are not linearly independent—selecting a basis from them seems difficult. However, we do have the

PROPOSITION 2. *The collections of manifolds $M(i_1, \cdots i_{k-1}, 0; s_1 \cdots s_{n-1})$ $i_1 \geqq i_2 \cdots i_{k-1}$; $s_1 \geqq s_2 \cdots s_{n-1}$; $k \geqq 2$ are linearly independent over $\mathfrak{N}_*$ and generate a direct summand of $\mathfrak{N}O(n)_*$.*

PROPOSITION 3. *All dependence relations among the generators are generated by relations involving a fixed k.*

The proof of these propositions involves an application of the spectral sequence of [4] for the group $O(n)$ and the representation $\rho_n \oplus \theta$ where ρ_n is the standard representation at $O(n)$ and θ is the trivial representation. In particular, we have

PROPOSITION 4. *There is a first quadrant spectral sequence $E^r_{p,q}$ whose E^1 term is given by*

$$\begin{aligned} E^1_{*,q} &= \sum_k \mathfrak{N}_*(BO(k) \times BO(q)) && 0 \leqq q < n, \\ &= \mathfrak{N}_*(BO(n)) && q = n, \\ &= 0 && q > n, \end{aligned}$$

and whose E^∞ term is associated to a filtration of $\mathfrak{N}O(n)_*$. Moreover, $d_1\colon E^1_{*,q} \to E^1_{*,q+1}$ is given by $d_1 = p_* \circ \pi^\#$ where $\pi^\#\colon \mathfrak{N}_*(BO(k) \times BO(q)) \to \mathfrak{N}_*(BO(k-1) \times BO(1) \times BO(q))$ is the bordism transfer homomorphism [1] associated to the natural projection $\pi\colon B(O(k-1) \times O(1) \times O(q)) \to B(O(k) \times O(q))$ and p_* is induced by $p\colon B(O(k-1) \times O(1) \times O(q)) \to B(O(k-1) \times O(q+1))$.

The computations are best done in cobordism. One notes that $d_1\colon \mathfrak{N}^*(B(O(k) \times O(q))) \to \mathfrak{N}^*(B(O(k+1) \times O(q-1)))$ is linear as an $\mathfrak{N}^*(BO(k+q))$ module map. Let $W_1 \cdots W_{k+q}$ be the cobordism Stiefel-Whitney classes of $BO(k+q)$ and $v_1, \cdots v_q$ the cobordism Stiefel-Whitney classes of $BO(q)$.

PROPOSITION 5. *$\mathfrak{N}^*(BO(k) \times BO(q))$ is a free finitely generated $\mathfrak{N}^*(BO(k+q))$ module with generators $\{v_1^{i_1} \cdots v_q^{i_q}\}$ where $i_j \geqq 0$ and $\sum i_j \leqq k$.*

Finally, we have

PROPOSITION 6. *Up to units*

$$\begin{aligned} d_1(V_1^{i_1} V_2^{i_2} \cdots V_q^{i_q}) &= 0 && \text{if } \textstyle\sum i_j < k, \\ &= \overline{V}_1^{j_2} \overline{V}_2^{j_3} \cdots \overline{V}_{q-1}^{j_q} && \text{if } \textstyle\sum i_j = k \end{aligned}$$

where $\overline{V}_i \in \mathfrak{N}^*(BO(q-1))$. Hence, the sequence

$$\mathfrak{N}^*(BO(k) \times BO(q)) \xrightarrow{d_1} \mathfrak{N}^*(BO(k+1) \times BO(q-1)) \xrightarrow{d_1} \mathfrak{N}^*(BO(k+2) \times BO(q-2))$$

is exact if $k \geqq 0$ and the spectral sequence collapses at the E^2 level.

Theorem 1 now follows quickly.

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