FACTORIZATION OF OPERATOR VALUED ENTIRE FUNCTIONS

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Let $W(z)$ be a complex valued entire function of exponential type with nonnegative values on the real axis. We call $W(z)$ *factorable* if

$$W(z) = A^*(z)A(z)$$

where $A(z)$ is an entire function whose restriction to the upper half-plane is an outer function. Here $A^*(z) = \bar{A}(\bar{z})$. Recall that an outer function in the upper half-plane is a function of the form

$$f(z) = C \exp\left(\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{1+tz}{t-z}\,\frac{\log k(t)}{1+t^2}\,dt\right), \qquad y > 0,$$

where C is a constant of absolute value 1, $k(t) \geqq 0$ a.e. on $(-\infty, \infty)$, and $(1+t^2)^{-1}\log k(t) \in L^1(-\infty, \infty)$. Of necessity, $k(x) = \lim |f(x+iy)|$ a.e. where the limit is taken as y decreases to zero. Therefore the restriction of an entire function $A(z)$ to the upper half-plane is an outer function if and only if $(1+t^2)^{-1}\log|A(t)| \in L^1(-\infty, \infty)$ and

$$\log|A(z)| = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\log|A(t)|}{(t-x)^2+y^2}\,dt, \qquad y > 0.$$

The following facts are available from the classical theory of entire functions:

(1°) for $W(z)$ to be factorable, it is necessary and sufficient that

$$\int_{-\infty}^{+\infty} \frac{\log^+ W(x)}{1+x^2}\,dx < \infty,$$

(2°) if $W(z)$ is factorable, the factor $A(z)$ is determined to within a multiplicative constant of absolute value 1,

(3°) if $W(z)$ is factorable, say $W(z) = A^*(z)A(z)$ as above, and if $W(z)$ is of exponential type τ, then $\exp(-\frac{1}{2}i\tau z)A(z)$ is of exponential type $\frac{1}{2}\tau$. See [2, p. 125], [3, p. 34], and [4, p. 437], where some original sources are cited. The purpose of this note is to communicate extensions of these results to operator valued entire functions.

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Let $\mathcal{C}$ be a separable complex Hilbert space. By a vector or operator valued function we shall mean a function whose values are vectors in $\mathcal{C}$ or bounded operators on $\mathcal{C}$ respectively. Analyticity is defined in the weak sense. The bar of a bounded operator on $\mathcal{C}$ denotes its adjoint, and we use the notation $A^*(z) = \overline{A}(\bar{z})$ for operator as well as scalar valued entire functions. By $H^2_{\mathcal{C}}$ we mean the Hardy space of vector valued analytic functions $f(z)$ defined for $y > 0$ such that

$$\|f\|^2 = \lim_{y\to 0} \int_{-\infty}^{+\infty} \|f(x+iy)\|^2_{\mathcal{C}}\, dx < \infty.$$

If $\mathcal{B}$ is a closed subspace of $\mathcal{C}$, $H^2_{\mathcal{B}}$ denotes the closed subspace of $H^2_{\mathcal{C}}$ of functions with values in $\mathcal{B}$. An operator valued analytic function $A(z)$ defined for $y > 0$ is called *outer* if there exists a bounded scalar valued outer function $f(z)$ such that

(i) $B(z) = f(z)A(z)$ is bounded for $y > 0$, and

(ii) the range of multiplication by $B(z)$ in $H^2_{\mathcal{C}}$ is dense in a subspace of the form $H^2_{\mathcal{B}}$.

In this case the closed subspace $\mathcal{B}$ of $\mathcal{C}$ implied in (ii) does not depend on the choice of $f(z)$, and we say that $A(z)$ acts in $\mathcal{B}$. A vector valued entire function $f(z)$ is said to be of exponential type τ, $\tau \geqq 0$, if for every vector c in $\mathcal{C}$ the scalar valued entire function $f_c(z) = \langle f(z), c\rangle_{\mathcal{C}}$ is of exponential type τ. An operator valued entire function $W(z)$ is said to be of exponential type if for every vector c in $\mathcal{C}$ the vector valued entire function $W(z)c$ is of exponential type $\tau(c)$ for some number $\tau(c) \geqq 0$.

Let $W(z)$ be an operator valued entire function of exponential type which has nonnegative values on the real axis. We call $W(z)$ *factorable* if

$$W(z) = A^*(z)A(z)$$

where $A(z)$ is an operator valued entire function whose restriction to the upper half-plane is an outer function. We can now state our main results. Let I denote the identity operator on $\mathcal{C}$.

THEOREM 1. *Let $W(z)$ be an operator valued entire function of exponential type which has nonnegative values on the real axis. Assume that there exists a scalar valued entire function $w(z)$ of exponential type such that*

$$W(x) \leqq w(x)I$$

for all real x. If $w(z)$ is factorable, then so is $W(z)$.

THEOREM 2. *Let $W(z)$ be an operator valued entire function of exponential type which has nonnegative values on the real axis. For each $j=1, 2$, let*

$$W(z) = A_j^*(z)A_j(z)$$

where $A_j(z)$ is an operator valued entire function whose restriction to the upper half-plane is an outer function, and let this outer function act in the subspace $\mathcal{B}_j$ of $\mathcal{C}$. Then

$$A_2(z) = UA_1(z)$$

where U is a partially isometric operator on $\mathcal{C}$ with initial set $\mathcal{B}_1$ and final set $\mathcal{B}_2$.

THEOREM 3. *Let $W(z)$ be an operator valued entire function of exponential type which has nonnegative values on the real axis. Assume that $W(z)$ admits a scalar dominant as in Theorem 1. Assume that the scalar dominant is factorable, and let*

$$W(z) = A^*(z)A(z)$$

where $A(z)$ is an operator valued entire function whose restriction to the upper half-plane is an outer function. If for some vector c in $\mathcal{C}$, $W(z)c$ is of exponenital type $\tau(c)$, $\tau(c) \geqq 0$, then $\exp(-\frac{1}{2}i\tau(c)z)A(z)c$ is of exponential type $\frac{1}{2}\tau(c)$.

Consider the case $\dim \mathcal{C} < \infty$. Let $W(z)$ be an entire function of exponential type which has nonnegative values on the real axis. If

$$\int_{-\infty}^{+\infty} \frac{\log^+[\operatorname{tr} W(x)]}{1 + x^2}\, dx < \infty,$$

then $W(z)$ is factorable. For in Theorem 1 we may choose $w(z) = \operatorname{tr} W(z)$.

Another corollary is valid for an arbitrary separable coefficient space $\mathcal{C}$. Let $W(z) = C_0 + C_1 z + \cdots + C_{2N} z^{2N}$ be a polynomial with operator coefficients which has nonnegative values on the real axis. Then

$$W(z) = A^*(z)A(z)$$

where $A(z) = A_0 + A_1 z + \cdots + A_N z^N$ is a polynomial with operator coefficients whose restriction to the upper half-plane is an outer function.

Proofs of these results will appear elsewhere. Theorem 2 is a special case of a more general assertion concerning outer functions. It is deduced from [6, Theorem 3]. In the case where $W(z)$ is bounded on the real axis, i.e. when we can choose $w(z)$ to be a positive constant, Theorems 1 and 3 are proved using a Hilbert space method originated by D. Lowdenslager [5] and developed by the first author [6]. The general cases of Theorems 1 and 3 are obtained from this special case by means of a theorem of A. Beurling and P. Malliavin [1].

References

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