ORDERS IN ARTINIAN RINGS

BY ARUN VINAYAK JATEGAONKAR

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The object of this note is to indicate some results concerning (right) skew polynomial rings over right orders in right Artinian rings. Detailed proofs will be published elsewhere.

We recall a few definitions first. A *right Artinian ring* is a ring with unity satisfying the descending chain condition on right ideals. A subring S of a right Artinian ring Q is called a *right order* in Q if every regular element in S is a unit in Q and every $q \in Q$ can be expressed as $q = sc^{-1}$, where $s, c \in S$.

Let R be a ring with unity and S be a unitary subring of R. Suppose there exists an element $x \in R$ such that every nonzero $r \in R$ can be uniquely expressed as

$$r = \sum_{i=0}^{k} x^{n_i} s_i$$

where s_i, $0 \leqq i \leqq k$, are nonzero elements of S and $0 \leqq n_0 < \cdots < n_k$ are integers. Further, suppose there exists a unitary monomorphism $\rho: S \to S$ such that $sx = x\rho(s)$ holds for every $s \in S$. In such a situation, we denote R as $S[x, \rho]$ and call it a right *skew polynomial ring* over S. If I is an ideal of S such that $\rho(I) \subseteq I$ then I is said to be *ρ-invariant*. We have

THEOREM 1. *Let S be a ring with unity which is a right order in a right Artinian ring and let $R = S[x, \rho]$. Then R is a right order in a right Artinian ring $\hat{Q}$. $\hat{Q}$ is semisimple if and only if S is semiprime. $\hat{Q}$ is simple if and only if S is semiprime and every nonzero ρ-invariant two-sided ideal of S is an essential right ideal of S.*

We need some more definitions. Let Q be a semisimple (Artinian) ring, $\{f_1, \cdots, f_m\}$ be the set of all the distinct central idempotents of Q which are primitive in the centre of Q and let $\rho: Q \to Q$ be a monomorphism. Then there exists a unique permutation σ on $\{1, \cdots, m\}$ such that $\rho(f_i) = f_{\sigma(i)}$ for $1 \leqq i \leqq m$. If $\sigma = \sigma_1 \cdots \sigma_k$ is a decomposition of σ into disjoint cycles then $\max_{1 \leqq i \leqq m}\{\text{length } \sigma_i\}$ is is called the *shuffling index* of ρ. Recall that a ring R has *right rank* m if m is the least integer such that every right ideal of R has a system of generators containing at most m elements.

THEOREM 2. *Let Q be a semisimple right Artinian ring and $R=Q[x,\rho]$ where ρ: $Q\to Q$ is a monomorphism of shuffling index m. Then R is a semiprime right hereditary right Noetherian ring of right rank m.*

Further details about the structure of $R=Q[x, \rho]$ are obtained.

We briefly indicate the main steps in the proof of Theorem 1. Firstly, we obtain the following analogue of Faith-Utumi theorem [1] which may be of some independent interest.

THEOREM 3. *Let S be a right order in a right Artinian ring Q. Then there exists a set $\{e_i: 1\leqq i\leqq n\}$ of orthogonal primitive idempotents in Q with $\sum_{i=1}^{n} e_i=1$ and there exist subgroups M_{ij} of e_iQe_j, $1\leqq i, j\leqq n$, such that $R= \bigoplus_{i,j=1}^{n} M_{ij}$ is a subring of S and a right order in Q. Further, each M_{ii} is a right order in the completely primary ring e_iQe_i.*

It follows from Theorem 3 that, if S is a right order in a right Artinian ring Q and if ρ: $S\to S$ is a monomorphism then it can be uniquely extended to a monomorphism ρ: $Q\to Q$; further, if $Q[x, \rho]$ is a right order in a right Artinian ring $\hat{Q}$ then $S[x, \rho]$ is also a right order in $\hat{Q}$. It then remains to show that $Q[x, \rho]$ is a right order in a right Artinian ring for an arbitrary right Artinian ring Q.

We firstly consider the case when Q is semisimple; in this case, if the shuffling index of ρ is m then $Q[x^m, \rho^m]$ is a semiprime principal right ideal ring and $Q[x, \rho]$ is a finitely generated right module over $Q[x^m, \rho^m]$. If $\mathfrak{D}$ denotes the set of all those right polynomials in $Q[x^m, \rho^m]$ which have a unit in Q as a leading coefficient, then $\mathfrak{D}$ is an exhaustive right divisor set in $Q[x, \rho]$. Cf. [4].

For the case of an arbitrary right Artinian ring Q, we show that every monomorphism ρ: $Q\to Q$ induces a monomorphism $\bar{\rho}$: $\overline{Q}\to\overline{Q}$ where $\overline{Q}=Q/J(Q)$. The lift of the exhaustive right divisor set $\mathfrak{D}$ in $\overline{Q}[\bar{x}, \bar{\rho}]$ is then shown to be an exhaustive right divisor set in $Q[x, \rho]$. This establishes that $\hat{Q}$ exists. Our arguments do not need nontrivial parts of the internal characterizations of right orders in semisimple rings or arbitrary right Artinian rings. Some special cases of Theorems 1 and 2 are known. Cf. [2], [3], [4].

REFERENCES

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4. ———, *Orders in Artinian rings*. II. J. Algebra **9** (1968), 266–273.

CORNELL UNIVERSITY, ITHACA, NEW YORK 14850