CHARACTERISTIC CLASSES–OLD AND NEW[1,2]

BY FRANKLIN P. PETERSON

1. Definition of sphere bundles. Let M^n be an n-dimensional, C^∞-manifold. Define $T(M)$ to be all vectors tangent to M of unit length. Define $p: T(M) \to M$ by $p(\text{vector}) = \text{initial point of the vector}$. Then p is a continuous function with $p^{-1}(m)$ homeomorphic to S^{n-1} if $m \in M$. $(T(M), p, M)$ is an example of an $(n-1)$-sphere bundle.

Let me now abstract some of the properties of this example and define an $(n-1)$-sphere bundle. An $(n-1)$-sphere bundle ξ is a triple (E, p, X), where $p: E \to X$ is a continuous function, X has a covering by neighborhoods $\{V_\alpha\}$ such that $h_\alpha: p^{-1}(V_\alpha) \to V_\alpha \times S^{n-1}$, where h is a homeomorphism, $h_\alpha(e) = (p(e), S_\alpha(e))$. That is, we can give coordinates to $p^{-1}(V_\alpha)$ using V_α and S^{n-1}. Furthermore, there is a condition on changing coordinates; namely, if $e \in p^{-1}(V_\alpha \cap V_\beta)$, then $h_\alpha(e) = (p(e), S_\alpha(e))$ and $h_\beta(e) = (p(e), S_\beta(e))$ and we obtain a function $S^\alpha_\beta: S^{n-1} \to S^{n-1}$ given by $S^\alpha_\beta(S_\alpha(e)) = S_\beta(e)$, defined for each $p(e) \in V_\alpha \cap V_\beta$. We demand that $S^\alpha_\beta \in O(n)$, the orthogonal group of homeomorphisms of S^{n-1}. Finally, S^α_β depends on $p(e)$ and this dependence must be continuous.

Two $(n-1)$-sphere bundles ξ and η over X are called equivalent if there is a homeomorphism $F: E_\xi \to E_\eta$ such that

$$\begin{array}{ccc} E_\xi & \xrightarrow{F} & E_\eta \\ & p\searrow \quad \swarrow p & \\ & X & \end{array}$$

commutes and such that $F \mid p^{-1}(x) \in O(n)$ for all coordinates on $p^{-1}(x)$.

A very important example of an $(n-1)$-sphere bundle is the following one. Let $\mathrm{BO}(n) =$ the Grassmann space of all n-planes through the origin in R^∞. Let $\mathrm{EO}(n)$ be the set of pairs, an element of $\mathrm{BO}(n)$ and a unit vector in that n-plane. Let $p: \mathrm{EO}(n) \to \mathrm{BO}(n)$ be the first element of the pair. The importance of this example is shown by the following classification theorem.

[1] An address delivered before the New York meeting of the Society by invitation of the Committee to Select Hour Speakers, April 13, 1968; received by the editors April 21, 1969.

[2] In order not to obscure the structure of the subject, I have left out a number of technicalities; in fact some of the statements may be incorrect as stated.

CLASSICAL CLASSIFICATION THEOREM. *The equivalence classes of* $(n-1)$*-sphere bundles over* X *are in one-to-one correspondence with the homotopy classes of maps of* X *into* $\mathrm{BO}(n)$.

2. **Definition of characteristic classes.** Roughly speaking, a characteristic class is a cohomology class in $H^*(X)$ assigned to a bundle ξ over X which is natural with respect to bundle maps. Rather than give a precise definition, let me give a construction.

Let $u\in H^*(\mathrm{BO}(n))$. Let ξ be an $(n-1)$-sphere bundle over X corresponding to a map $f_\xi\colon X\to \mathrm{BO}(n)$ by the above theorem. u defines a characteristic class $u(\xi)\in H^*(X)$ by $u(\xi)=f_\xi^*(u)$, where $f_\xi^*\colon H^*(\mathrm{BO}(n))\to H^*(X)$ is the homomorphism induced by f_ξ (recall that f_ξ^* depends only on the homotopy class of f_ξ).

Thus, to study characteristic classes, we must study $H^*(\mathrm{BO}(n))$. The answers, with various fields for coefficients, are as follows.[3] $H^*(\mathrm{BO}(n); Z_2)$ is a polynomial ring over Z_2 on generators $W_i\in H^i(\mathrm{BO}(n); Z_2)$, $i=1, \cdots, n$. $W_i(\xi)=f_\xi^*(W_i)$ is called the ith Stiefel-Whitney class of ξ. $H^*(\mathrm{BO}(\infty); Z_p)$ and $H^*(\mathrm{BO}(\infty); Q)$ are polynomial rings over Z_p (p is an odd prime) and Q respectively on generators $P_i\in H^{4i}(\mathrm{BO}(\infty); Z_p)$ or $H^{4i}(\mathrm{BO}(\infty); Q)$, $i=1, \cdots$, $P_i(\xi)=f_\xi^*(P_i)$ is called the ith Pontrjagin class of ξ.

3. **Some examples of applications of characteristic classes.** The study of characteristic classes has been very useful in differential geometry, differential topology, and algebraic topology. I will now give a few examples of such applications.

I. *Cobordism.* Let M^n be a closed, connected, C^∞-manifold of dimension n. Then $M^n=\partial W^{n+1}$, where W^{n+1} is a compact, connected, C^∞-manifold with boundary, if and only if $f_\tau^*\colon H^n(\mathrm{BO}(n); Z_2)\to H^n(M^n; Z_2)$ is zero where τ is the tangent bundle described at the beginning of this lecture [**17**].

II. *Homotopy spheres.* Let Θ^n be the group of diffeomorphism classes of homotopy spheres. Pontrjagin classes have been used to study these groups. For example, $\Theta^{15}\approx Z_{8128}$ [**6**].

III. *Embeddings and immersions.* Given M^n, the problem is to find the smallest k such that M^n can be differentiably embedded or immersed in R^{n+k}. The initial results were proved using Stiefel-Whitney classes. The techniques now are quite complicated and we are now near to solving this problem for real and complex projective spaces.

IV. *K-theory.* Let $\mathrm{KO}(X)$ be the set of equivalence classes of bundles over X, with dimension $X<n$. This forms a group and acts

[3] An excellent introduction to the classical theory, including proofs of the following assertions, is to be found in [**10**].

as a cohomology theory which has turned out to be very useful. For example, one can prove that the maximum number of linearly independent tangent vector fields on S^{n-1} is 2^c+8d-1, where $n=(2q+1)2^b$, $b=c+4d$, $0 \leqq c \leqq 3$ [1].

4. **More general bundles.** In recent years it has become clear that one should study $(n-1)$-sphere bundles where the changes of coordinates can be allowed to be in larger groups than $O(n)$. Examples of such groups, in increasing size, are: PL(n) = piecewise linear homeomorphisms of S^{n-1}, Top(n) = homeomorphisms of S^{n-1}, and G(n) = homotopy equivalences of S^{n-1}.

NEW CLASSIFICATION THEOREM. *Let $H=PL$, Top, or G. There exists a space $BH(n)$ such that the equivalence classes of $(n-1)$-sphere bundles with group H over X are in one-to-one correspondence with the homotopy classes of maps of X into $BH(n)$* ([11] *and* (14]).

Using this theorem, we can give the same construction of characteristic classes as we did in the classical case. In order to use these characteristic classes, we need to know $H^*(\mathrm{BH}(n))$ with various coefficients. Most of the rest of this paper is devoted to describing what is known about $H^*(\mathrm{BH})$, where $\mathrm{BH}=\lim_{n\to\infty} \mathrm{BH}(n)$.

5. $\pi_*(\mathrm{BH})$. Before stating the results on cohomology, let me first give the known results on the homotopy groups of the classifying spaces.

I. $\pi_*(\mathrm{BO})$ is periodic of period 8 with $\pi_{8k+i}(\mathrm{BO})=Z, Z_2, Z_2, 0, Z, 0, 0, 0$, with $i=0, 1, 2, 3, 4, 5, 6$, and 7 respectively [2].

II. $0\to\pi_i(\mathrm{BO})\to\pi_i(\mathrm{BPL})\to\Gamma_{i-1}\to 0$ is an exact sequence where Γ_{i-1} is a finite group which is partially known. Also, the structure of the exact sequence is known ([5] and [4]).

III. $\pi_i(\mathrm{Top/PL})=0$ if $i\neq 3$, and $\pi_3(\mathrm{Top/PL})=Z_2$ [7].

IV. $\pi_i(\mathrm{BG})=\pi_{i-1+k}(S^k)$, k large; hence, known to a certain extent.

V. $\pi_i(\mathrm{G/PL})$ is periodic of period 4 with $\pi_{4k+i}(\mathrm{G/PL})=Z, 0, Z_2, 0$, with $i=0, 1, 2$, and 3 respectively [15].

6. $H^*(\mathrm{BH}; Q)$. Using the above results on π_*, it is easy to see that $H^*(\mathrm{BG}; Q)=0$ if $i>0$ and that $H^*(\mathrm{BTop}; Q)\to H^*(\mathrm{BPL}; Q)\to H^*(\mathrm{BO}; Q)\to Q[P_1, \cdots]$ are all isomorphisms.

7. $H^*(\mathrm{BH}; Z_2)$. There exists a connected Hopf algebra $C(H)$ over the mod 2 Steenrod algebra A_2 such that $H^*(\mathrm{BH}; Z_2)\approx H^*(\mathrm{BO}; Z_2)\otimes C(H)$, as Hopf algebras over A_2 [3]. $C(0)$ is trivial of course. $C(G)$ is 2-connected, and its structure has been determined recently [9]. $C(\mathrm{PL})$ and $C(\mathrm{Top})$ are still unknown.

8. $H^*(\mathrm{BH}; Z_p)$, p **odd.** The situation is a little different from the case $p=2$. Analogous to the case $p=2$ we have $H^*(\mathrm{BG}; Z_p) \approx (Z_p[q_i]\otimes E(\beta q_i))\otimes C_p(\mathrm{G})$, where $q_i \in H^{i(2p-2)}(\mathrm{BG}; Z_p)$ is the Wu class, βq_i is its Bockstein, and $C_p(\mathrm{G})$ is a Hopf algebra over A_p [**13**]. Furthermore, $C_p(\mathrm{G})$ is $(p(2p-2)-2)$-connected and its complete structure has been found very recently [**7**].

$H^*(\mathrm{BTop}; Z_p)\to H^*(\mathrm{BPL}; Z_p)$ is an isomorphism by §5, III, so we need only study $H^*(\mathrm{BPL}; Z_p)$. It is an unpublished theorem that if p is an odd prime, BPL is of the same mod p homotopy type as $\mathrm{BO}\times B$ Coker J, where B Coker J is a space whose homotopy groups are the cokernel of the homeomorphism J: $\pi_*(\mathrm{BO})\to\pi_*(\mathrm{BG})$ [**16**]. However, the map $\mathrm{BO}\times\mathrm{pt.}\to\mathrm{BO}\times B$ Coker $J\to$BPL is not the usual map so this is quite different from the case $p=2$. Also, the map J_{PL}: BPL$\to$BG has the property that $J^*_{\mathrm{PL}}(\beta q_i)=0$ if $i\leqq p$ and is not zero if $i\geqq p+1$. The best conjecture at present seems to be that $C_p(\mathrm{G})\approx H^*(B$ Coker J; $Z_p)$. To complete the picture, we need to know $J^*_{\mathrm{PL}}(q_i)$ explicitly [**12**].

9. **Applications.** One expects that a good knowledge of these new characteristic classes will lead to many applications as in the classical case. I mention only one, namely that §3, I generalizes to the PL case. That is, let M^n be a closed, connected PL-manifold. Then $M^n=\partial W^{n+1}$, where W^{n+1} is a compact, connected PL-manifold with boundary, if and only if f_τ^*: $H^n(\mathrm{BPL}; Z_2)\to H^n(M^n; Z_2)$ is zero [**3**]. A similar theorem is true for oriented C^∞-manifolds, but for oriented PL-manifolds, it fails in dimension 27 (though true in lower dimensions) [**12**].

Bibliography

1. J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632.

2. R. Bott, *The stable homotopy of the classical groups*, Ann of Math. (2) **70** (1959), 313–337.

3. W. Browder, A. Liulevicius and F. P. Peterson, *Cobordism theories*, Ann. of Math. (2) **84** (1966), 91–101.

4. G. Brumfiel, *On the homotopy groups of BPL and PL/O*, Ann. of Math. (2) **88** (1968), 291–311.

5. M. Hirsch and B. Mazur, *Smoothings of piecewise linear manifolds*, Mimeographed notes, University of Cambridge, Cambridge, England, 1964.

6. M. Kervaire and J. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. (2) **77** (1963), 504–537.

7. R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc. **75** (1969), 742–749.

8. J. P. May (to appear).

9. R. J. Milgram (to appear).

10. J. W. Milnor, *Lectures on characteristic classes*, Mimeographed notes, Princeton University, Princeton, N. J., 1957.

11. ———, *Microbundles*. I, Topology **3** (1964), suppl. 1, 53–80.

12. F. P. Peterson, *Some results on PL-cobordism* (to appear).

13. F. P. Peterson and H. Toda, *On the structure of* H^* $(BSF; Z_p)$, J. Math. Kyoto Univ. **7** (1967), 113–121.

14. J. D. Stasheff, *A classification theorem for fibre spaces*, Topology **2** (1963), 239–246.

15. D. Sullivan (to appear).

16. ——— (to appear).

17. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1964), 17–86.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139