A "FUNCTIONAL EQUATION" FOR MEASURES AND A GENERALIZATION OF GAUSSIAN MEASURES

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1. **Introduction.** Let G be a LCA group for which the map $x \mapsto 2x$ is an automorphism, and let $\xi: G\times G \to G\times G$ be defined by $\xi(x, y) = (x+y, x-y)$. We call a regular complex-valued measure μ on G Gaussian iff $\exists$ a second measure ν on G such that for all Borel sets $E \subseteq G\times G$,

$$(\mu \times \mu)(E) = (\nu \times \nu)(\xi(E)). \tag{1.1}$$

One rationale for this definition is that any finite probability measure on $\boldsymbol{R}$ which satisfies (1.1) is a Gaussian distribution with mean 0. (See [1, p. 77] for a proof.) Another reason is that the 2-adic theta functions defined by Mumford in [2] are related to 2-adic measures satisfying (1.1) much as ordinary theta functions are related to the Gaussian distribution $\exp(-ax^2)dx$.

Actually, we shall consider all set functions which are finite complex linear combinations of regular measures on G. These need not be σ-additive measures (since the regular measures need not be bounded), but we shall use the term measure for such functions as well.

The problem we consider is that of determining all Gaussian measures on G. In [2], Mumford did this in the case $G=(\boldsymbol{Q}_2)^n$; in §§3 and 4 of this paper we state the results for $G=\boldsymbol{R}^n$ and for G a compact group. One reason for considering these cases is given by the following structure theorem.

THEOREM 1. *If G is a LCA group such that $x\mapsto 2x$ is an automorphism, then G can be written as $V\times W\times G_0$, where V is a real vector group, W is a 2-adic vector group, and G_0 contains a compact open subgroup for which $x\mapsto 2x$ is an automorphism.*

Another attack on the problem is considered in §2, where we consider Gaussian measures which are absolutely continuous (with respect to Haar measure). The rationale behind this approach is the following result.

[1] The results announced here are contained in the author's Ph.D. thesis at Harvard University, written while he held an N.S.F. Graduate Fellowship.

THEOREM 2. *If μ is Gaussian, then μ is either absolutely continuous or singular.*

However, the singular Gaussian measures are in general hard to analyze.

2. The absolutely continuous case. Let μ, ν, be absolutely continuous measures on G satisfying (1.1), and let f, g be their respective Radon-Nikodým derivatives. Then f and g satisfy

$$f(x)f(y) = g(x + y)g(x - y) \quad \text{a.e. on } G \times G. \tag{2.1}$$

THEOREM 3. *Let f, g be nonnull functions satisfying* (2.1). *Then there is an open subgroup G_0 of G, closed under division by* 2, *such that f and g are nonzero a.e. on G_0 and are zero a.e. off G_0. Moreover, there are continuous functions f_0, g_0: $G_0 \to C^x$, and a complex number c_0, such that:*

(a) *$c_0 f_0 = f$, $\pm c_0 g_0 = g$ a.e. on G, for some fixed choice of sign.*

(b) *f_0 and g_0 are quadratic characters (i.e., $f_0(x+y)/f_0(x)f_0(y)$ is bilinear in x and y, and similarly for g_0).*

It follows from continuity that f_0 and g_0 satisfy (2.1) everywhere on G_0 and hence that $g_0(x)^2 = f_0(x)$.

To show that G_0 exists is fairly straightforward; in what follows, we assume that $G_0 = G$. If (2.1) held everywhere, and f, g were never 0, we would proceed as follows: let $c_0 = f(0)$, and let $f_0 = f/c_0$, $g_0 = g/\pm c_0$, where the sign is chosen so that $g_0(0) = 1$. Then f_0 and g_0 satisfy (2.1). Let $y = 0$; we get $g_0(x)^2 = f_0(x)$; now let $x = 0$ to show that $g_0(y) = g_0(-y)$. Hence g_0 and f_0 are even functions. Substituting in (2.1), we find that

$$f_0(x + y)f_0(x - y) = f_0(x)^2 f_0(y)^2. \tag{2.2}$$

This is essentially the parallelogram law; it follows without much trouble that $f_0(x+y)/f_0(x)f_0(y)$ is bilinear.

In the actual theorem, substituting specific values for x and y in (2.1) is invalid. Instead, we use limit arguments, based on the density theorem and Lusin's theorem, to get the result.

3. Gaussian measures on compact groups. We begin by reducing the problem to a special case. Let f be the Fourier-Stieltjes transform of the Gaussian measure μ; an easy argument shows that f has support on a subgroup $\Gamma_0 \subseteq G^\wedge$ closed under division by 2 and that f is a multiple of a quadratic character on Γ_0. We may assume that $f(0) = 1$. Set $G_0 = \Gamma_0^\wedge$. Then there is a Gaussian measure μ_0 on G_0 (which we shall call the condensation of μ) whose Fourier-Stieltjes transform is $f \uparrow \Gamma_0$. It is easy to obtain either μ or μ_0 in terms of the other; thus for convenience, we shall assume that $\Gamma_0 = G^\wedge$.

Suppose that $G^\wedge$ is a torsion-free group of rank n. (Since $x \mapsto 2x$ is an automorphism of G, $G^\wedge$ is a $\boldsymbol{Z}[\frac{1}{2}]$-module.) Then we can define a Gaussian measure on G as follows: G contains a dense image of $\boldsymbol{R}^n$, and any finite Gaussian measure on $\boldsymbol{R}^n$ therefore defines a Gaussian measure on G. Let B be any symmetric complex matrix whose real part is positive definite, and let

$$d\mu_B(x) = \frac{1}{(\mathrm{Det}\ B)^{1/2}} \exp((\pi B^{-1}x, x))dx.$$

(Here, (,) is the usual scalar product, and the sign of B must be chosen so that $\mu_B(\boldsymbol{R}^n) = 1$.) We call such a measure on G a matrix measure. More generally, we call a Gaussian measure μ on G matricial iff

(a) $\mu^\wedge$ is never 0;

(b) μ is the weak-$*$ limit of Gaussian measures μ whose Fourier-Stieltjes transforms are concentrated on subgroups H of finite rank and whose condensations are matrix measures on $H^\wedge$.

The opposite extreme from a matricial measure is a large measure. We say that the Gaussian measure μ is large iff $|f(x)| = 1$ for all $x \in G^\wedge$. A more useful characterization is the following

THEOREM 4. *μ is a large Gaussian measure $\Leftrightarrow \mu$ is a Gaussian point measure concentrated on a finite subgroup of G.*

We can now state the characterization of Gaussian measures on G.

THEOREM 5. *Let μ be a Gaussian measure on G whose Fourier transform never vanishes. Then there are measures μ_1 and ν, and a subgroup G_1, such that:*

(1) *μ_1 is concentrated on G_1; as a measure on G_1, μ_1 is matricial;*

(2) *ν is a large measure;*

(3) $\mu_1 * \nu = \mu$.

Moreover, (1), (2), *and* (3) *uniquely determine μ_1, ν, and G_1.*

The proofs of these theorems involve getting restrictions on f in terms of $\|\mu\|$, then turning around and showing that if f satisfies the appropriate restrictions, we can actually construct μ.

4. **The reals.** Let μ be a Gaussian measure on $\boldsymbol{R}^n$. Then supp μ is a subspace of $\boldsymbol{R}^n$; we may therefore assume supp $\mu = \boldsymbol{R}^n$.

THEOREM 4.1. *There is a symmetric nonsingular complex $n\times n$ matrix B' such that $d\mu(x) = \exp(\pi Bx, x)dx$. (Here, dx is Haar measure.) In particular, μ is absolutely continuous.*

The proof amounts to showing that $\exp(-nx,x)$ is μ-integrable for large enough n. After that, a Fourier transform argument gives the rest.

Arguments similar to those of Theorem 4.1 can be used to find all Gaussian measures on $R^n\times C$, where C is an arbitrary compact group.

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