A NEW CHARACTERIZATION OF $AR(\mathfrak{M})$ AND $ANR(\mathfrak{M})$

BY NEIL GRAY

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The $AR(\mathfrak{M})$ spaces have been characterized as the r-images of convex sets in normed linear spaces, and the $ANR(\mathfrak{M})$ spaces as the r-images of open subsets of such sets. (The r-images of a space correspond to the retracts of the space. See Borsuk [1] for proofs of most of the results referred to here.) In this paper we present new characterizations of these spaces. The characterizations yield some interesting new results, as well as clarifications and new proofs of older results.

1. **The characterizations.** The fundamental notion is that of a guiding function, first defined (in the case of separable spaces) by Wojdyslawski [3].

DEFINITIONS. A *guiding function* for a metrizable space X is a continuous mapping g from a CW-polytope P into X such that

(i) every finite set S of vertices of P determines a simplex (denoted by conv$S$) in P;

(ii) g maps the vertices of P onto a dense subset of X;

(iii) if each S_n, $n=1, 2, \cdots$ is a finite set of vertices of P and $\lim_{n\to\infty} g(S_n)=x$ then $\lim_{n\to\infty} g(\mathrm{conv}S_n)=x$.

A *locally guiding function* for a metrizable space is a mapping g which satisfies every condition for a guiding function except (i), which is replaced by

(i)′ Every $x\in X$ has a neighborhood W_x with the following property: every finite set S of vertices of P such that $g(S)\subset W_x$ determines a simplex conv$S$ in P.

THEOREM 1. *A metrizable space is an $AR(\mathfrak{M})$ if and only if it has a guiding function.*

THEOREM 2. *A metrizable space is an $ANR(\mathfrak{M})$ if and only if it has a locally guiding function.*

For a proof of Theorem 1 in the case of separable metric spaces see [3] and [2]. In the more general case a guiding function for an $AR(\mathfrak{M})$ space X is constructed as follows. We may assume X is a closed subset of a convex set K in a normed linear space and $r\colon K\to X$ is a retraction. If Z is any dense subset of X then in the real linear space R^Z we pick out the linearly independent set $V=\{v_z\in R^Z \mid z\in Z\}$, where $v_z(z')$ is 1 if $z=z'$ and 0 otherwise. Let $P=\mathrm{conv}\,V$ (convex hull of V) and give P the CW topology. Now define g_0 on V by $g_0(v_z)=z$ and extend g_0 linearly so as to be defined on every simplex in P. Then $g_0(P)\subset K$ and we define the guiding function $g\colon P\to X$ by $g=rg_0$.

2. **Applications.** The next four theorems illustrate the use of the above characterizations. Detailed proofs of these and of related theorems will appear elsewhere.

THEOREM 3. *A sufficient condition for a metric space X to be an $AR(\mathfrak{M})$ is that it be the continuous image of a convex subset K of a normed linear space under a mapping f which has the following property: for each $\epsilon>0$ there is a $\delta>0$ such that whenever A is a finite subset of K and* $\operatorname{diam}[f(A)]<\delta$ *then* $\operatorname{diam}[f(\operatorname{conv}A)]<\epsilon$.

PROOF. If Z is a dense subset of K and $g_0\colon P\to K$ is defined as above then it is a guiding function for K and $g=fg_0$ is a guiding function for X.

If X is the union of a collection of $AR(\mathfrak{M})$ subsets is X an $AR(\mathfrak{M})$? It is easy to show that if the collection consists of just two members then a sufficient condition for X to be an $AR(\mathfrak{M})$ is that these sets be closed and that their intersection is an $AR(\mathfrak{M})$. Also, there is an obvious analogy for any finite collection (all intersections must yield $AR(\mathfrak{M})$ spaces). For an infinite collection the situation is quite different. It is easy to find examples of a space X which is the union of an expanding sequence of closed subsets, each of which is an $AR(\mathfrak{M})$, where X is not even locally connected (and hence not an $ANR(\mathfrak{M})$). However, we do have

THEOREM 4. *If the metrizable space X is the union of an expanding sequence $\{A_n\}$ of closed subsets, each of which is an $AR(\mathfrak{M})$, and if $A_n\subset$ interior A_{n+1} for every n, then X is an $AR(\mathfrak{M})$.*

A subset A of a compact metric space (X, d) is strongly convex (written SC) if to every pair of points $x, y\in A$ and each $t\in[0\colon 1]$ there corresponds a unique point $z\in A$ such that $d(x, z)=td(x, y)$ and $d(z, y)=(1-t)d(x, y)$. We then write $z=ty+(1-t)x$. A metric space is locally strongly convex (written loc SC) if every point has a basis consisting of SC neighborhoods. It is easy to see that a compact SC metric space is contractible and locally contractible and hence, if finite dimensional, is an $AR(\mathfrak{M})$. Similarly, a finite dimensional compact metric space in which every point has at least one SC neighborhood is an ANR$(\mathfrak{M})$. It would be interesting to know if one can drop the condition of finite dimensionality in either of these cases.

THEOREM 5. *A locally compact metric space which is loc SC is an* $ANR(\mathfrak{M})$.

THEOREM 6. *A compact metric space which is both SC and loc SC is an* $AR(\mathfrak{M})$.

The proof of Theorem 5 yields a proof of Theorem 6 because a contractible $ANR(\mathfrak{M})$ is an $AR(\mathfrak{M})$.

PROOF OF THEOREM 5. We construct a locally guiding function for X. Let Z be a dense subset of X and construct the CW polytope P as in the introduction. We shall define a locally guiding function on P by induction on the dimension of its simplexes. As before, define $g(v_z)=z$ for each $z\in Z$. For each $x\in X$ let W_x denote a compact SC neighborhood of x. For each $n=1, 2, \cdots$ let $\mathcal{S}_n$ denote the collection of n-simplexes conv$S$ such that $g(S)\subset W_x$ for some x. We extend g to the simplexes in $\mathcal{S}_1$ as follows: for any conv$\{v_z, v_{z'}\}$ in $\mathcal{S}_1$ we define $g(tv_z+(1-t)v_{z'})=tz+(1-t)z'$, where the operations on the right-hand side are as defined previously. Now supposing that g has been defined for every $\mathcal{S}_n$, $n<k$, we define g on the simplexes of $\mathcal{S}_k$ as follows: for any simplex Δ in $\mathcal{S}_k$ we have $g(\text{vertices } \Delta)\subset W_x$ for some x, so g has already been defined on all the proper faces of Δ. Let p be the barycenter of Δ, and v_z be any one of its vertices, and let us set $g(p)=z$. Each point q of Δ which is different from p has a unique representation $q=tp+(1-t)q'$, where q' lies in a proper face of Δ. We define $g(q)=tg(p)+(1-t)g(q')$. The compactness and strong convexity of W_z together imply that g is well defined and continuous on Δ. Continuing this process we define a locally guiding function $g: P\to X$.

REFERENCES

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WESTERN WASHINGTON STATE COLLEGE, BELLINGHAM, WASHINGTON 98225