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TWO SIDED IDEALS OF OPERATORS

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1. Let X be a Banach space, and $B(X)$ the Banach algebra of all bounded linear operators in X. The closed two sided ideals of $B(X)$ (actually, of *any* Banach algebra) form a complete lattice $L(X)$. Aside from very concrete cases, $L(X)$ has not yet been determined; for instance, when $X=l^p$, $1\leqq p<\infty$, $L(X)$ is a chain (i.e., totally ordered) with three elements: $\{0\}$, $B(X)$ and the ideal $C(X)$ of compact operators (see [3]). On the other hand, it is known [2, 5.23] that for $X=L^p$, $1<p<\infty$, the lattice $L(X)$ is *not* a chain. A treatment for X a Hilbert space of arbitrary dimension can be found in [4]. We aim to exhibit here a Banach space X such that $L(X)$ is both "long" and "wide." Precisely, we have

PROPOSITION. *There exists a real Banach space X with the properties:*
(i) *X is separable, isometric to its dual X^*, and reflexive;*
(ii) *it is possible to assign a closed two sided ideal $\mathfrak{a}(\mathfrak{F})\subset B(X)$ to each finite set of positive integers $\mathfrak{F}$, in such a way that the mapping $\mathfrak{F}\rightarrow\mathfrak{a}(\mathfrak{F})$ is injective and inclusion preserving in both directions: $\mathfrak{F}\subseteq\mathfrak{G}$ if and only if $\mathfrak{a}(\mathfrak{F})\subseteq\mathfrak{a}(\mathfrak{G})$.*

The example is described below, in §3.

2. In the sequel, all Banach spaces are *real* (the complex case can be dealt with similarly). If X, Y are Banach spaces, $\mathfrak{m}(Y, X)$ denotes the set of operators $T\in B(X)$ that can be factorized through Y, i.e., such that $T=SQ$ for suitable bounded linear operators $Q: X\rightarrow Y$, $S: Y\rightarrow X$. If Y is isomorphic (as a Banach space) to its square $Y\times Y$ ($\times$ means cartesian product), then (see [6, Proposition 1.2] or [2, Theorem 5.13]) $\mathfrak{m}(Y, X)$ is a two sided ideal of $B(X)$. $\mathfrak{a}(Y, X)$ will denote the (uniform) closure of $\mathfrak{m}(Y, X)$; thus, if Y is isomorphic to $Y\times Y$, $\mathfrak{a}(Y, X)$ is a *closed two sided ideal* of $B(X)$.

In all that follows, *subspace* means closed lineal subspace; a subspace Y of a Banach space X is *complemented* if $X = Y+Z$ for some subspace Z satisfying $X\cap Z = \{0\}$.

We shall need the following generalization of Theorem 5.20, in [2].

LEMMA 1. *Let X be a Banach space, and Y a complemented subspace of X, isomorphic to its square $Y\times Y$. Then, for an arbitrary Banach space Z, the following conditions are equivalent:*

(i) $\mathfrak{m}(Y, X)\subseteq\mathfrak{a}(Z, X)$,

(ii) *Y is isomorphic to a complemented subspace of Z.*

PROOF. Let $P\in B(X)$ be a projection on Y (i.e., $P^2=P$, $PX=Y$), I: $Y\to Y$ the identity and J: $Y\to X$ the canonical injection; it is clear that $P\in\mathfrak{m}(Y, X)$. Let ϵ be a positive real number such that $\epsilon\|P\|<1$. Suppose now that $\mathfrak{m}(Y, X)\subseteq\mathfrak{a}(Z, X)$. There exist thus S: $Z\to X$, Q: $X\to Z$ such that $\|P-SQ\|<\epsilon$. Consider the operator $U\in B(Y)$ defined by $U=I-PSQJ$; since $I=PJ$, we see that $U=PJ-PSQJ =P(P-SQ)J$, and therefore

$$\|U\| \leqq \|P\| \; \|P - SQ\| \; \|J\| \leqq \|P\|\epsilon < 1.$$

Hence $PSQJ$: $Y\to Y$ is *invertible*, that is, there exists $T\in B(Y)$ such that $I=TPSQJ=VW$, where $V=TPS$: $Z\to Y$ and $W=QJ$: $Y\to Z$. This means that $I\in\mathfrak{m}(Z, Y)$, and from [6, Lemma 1.I] (or [2, 5.12]), we conclude that Y is isomorphic to a complemented subspace of Z, as desired. The converse is obvious: if Y' is a complemented subspace of Z isomorphic to Y, then $\mathfrak{m}(Y, X)=\mathfrak{m}(Y', X)\subseteq\mathfrak{m}(Z, X)\subseteq\mathfrak{a}(Z, X)$.

LEMMA 2. *Assume that X, Y_1, Y_2, $\cdots$, Y_n are Banach spaces such that Y_j is isomorphic to $Y_j\times Y_j$ for $j=1, 2, \cdots, n$. Then $\mathfrak{m}(Y_1, X) +\mathfrak{m}(Y_2, X)+ \cdots +\mathfrak{m}(Y_n, X)=\mathfrak{m}(Y_1\times \cdots \times Y_n, X)$.*

PROOF. An inductive argument reduces the proof to the case $n=2$, which is disposed of as follows. Since Y_1 and Y_2 are (isomorphic to) complemented subspaces of $Y_1\times Y_2$, it is clear that $\mathfrak{m}(Y_1, X) \subseteq\mathfrak{m}(Y_1\times Y_2, X)$ and $\mathfrak{m}(Y_2, X)\subseteq\mathfrak{m}(Y_1\times Y_2, X)$, whence

$$\mathfrak{m}(Y_1, X) + \mathfrak{m}(Y_2, X)\subseteq\mathfrak{m}(Y_1 \times Y_2, X).$$

Conversely, if $T=SQ\in\mathfrak{m}(Y_1\times Y_2, X)$, where S: $Y_1\times Y_2\to X$ and Q: $X\to Y_1\times Y_2$ with $Q(x)=(Q_1(x), Q_2(x))$, then we define S_1: $Y_1\to X$, S_2: $Y_2\to X$ as

$$S_1(y) = S(y, 0), \qquad S_2(y) = S(0, y);$$

finally, let T_1, $T_2\in B(X)$ be the operators $T_1=S_1Q_1$, $T_2=S_2Q_2$. Clearly $T_1+T_2=T$ with $T_j\in\mathfrak{m}(Y_j, X)$, $j=1, 2$, and therefore $T\in\mathfrak{m}(Y_1, X) +\mathfrak{m}(Y_2, X)$; the lemma follows.

Also, from [6, Lemma 1.I] (or [2, 5.12]) and [1, Theorem 7, p. 205], we obtain that for $p \neq q$, $p \geqq 1$, $q \geqq 1$, the ideal $\mathfrak{m}(l^q, l^p)$ is not the whole of $B(l^p)$. Since the ideal $C(l^p)$ of compact operator is the largest proper two sided ideal of $B(l^p)$ (see [3, Theorem 5.1]), it follows that

LEMMA 3. *If $p, q \geqq 1$, $p \neq q$, then $\mathfrak{m}(l^q, l^p) \subseteq C(l^p)$.*

3. Let $\mathcal{P}$ be a countable set of real numbers $p \geqq 1$; define Y as the product $Y = \Pi\{l^p;\ p \in \mathcal{P}\}$, where l^p is the ordinary (real) sequence space. We denote by $|x|$ the norm of an element $x \in l^p$, for all p. Consider now the set $l(\mathcal{P})$ of all families $\{x_p \in l^p;\ p \in \mathcal{P}\} \in Y$ such that $\sum\{|x_p|^2, p \in \mathcal{P}\} < \infty$ (this is always the case, if $\mathcal{P}$ is finite). It can be seen that $l(\mathcal{P})$ is a linear subspace of Y and that the norm $\|\{x_p\}\| = (\sum |x_p|^2)^{1/2}$ makes $l(\mathcal{P})$ a separable Banach space; if $\mathcal{P}$ is finite, $l(\mathcal{P}) = \prod\{l^p;\ p \in \mathcal{P}\}$. It is clear that for each subset $\mathcal{Q} \subset \mathcal{P}$, the space $l(\mathcal{Q})$ can be identified to a complemented subspace of $l(\mathcal{P})$. Moreover, $l(\mathcal{P})$ is always isomorphic to its square $l(\mathcal{P}) \times l(\mathcal{P})$ (see for instance [5, Proposition 3, b]). The dual $(l(\mathcal{P}))^*$ of $l(\mathcal{P})$ can be identified to $l(\mathcal{P}^*)$, where $\mathcal{P}^*$ is the set of conjugates p^* of elements $p \in \mathcal{P}$, i.e., $1/p + 1/p^* = 1$. In particular, if $1 \notin \mathcal{P}$, then $l(\mathcal{P})$ is *reflexive*, and furthermore, if $\mathcal{P} = \mathcal{P}^*$, then $l(\mathcal{P})$ is *isometric* to its dual. Therefore, such $l(\mathcal{P})$ satisfy condition (*i*) in the Proposition above.

Let $\mathcal{P}$ be a *fixed countably infinite set of real numbers $p > 1$ such that* $\mathcal{P} = \mathcal{P}^*$, and let X denote the space $X = l(\mathcal{P})$. For each finite subset $\mathfrak{F} \subseteq \mathcal{P}$, let $\mathfrak{a}(\mathfrak{F}) \subseteq B(X)$ be the ideal $\mathfrak{a}(\mathfrak{F}) = \mathfrak{a}(l(\mathfrak{F}), X)$. Since $l(\mathfrak{F})$ is (isomorphic to) a complemented subspace of $l(\mathfrak{G})$, whenever $\mathfrak{F} \subseteq \mathfrak{G}$ it is clear (Lemma 1) that the mapping $\mathfrak{F} \rightarrow \mathfrak{a}(\mathfrak{F})$ is inclusion preserving. On the other hand, suppose that $\mathfrak{a}(\mathfrak{F}) \subseteq \mathfrak{a}(\mathfrak{G})$, or, equivalently, $\mathfrak{m}(l(\mathfrak{F}), X) \subseteq \mathfrak{a}(l(\mathfrak{G}), X)$.

By Lemma 2, this inequality is equivalent to

$$\mathfrak{m}(l^p, X) \subseteq \mathfrak{a}(l(\mathfrak{G}), X), \qquad \text{for all } p \in \mathfrak{F}.$$

Lemma 1 applies, and we conclude that for $p \in \mathfrak{F}$, l^p is isomorphic to a complemented subspace of $l(\mathfrak{G})$. By [6, Lemma 1.I] (or [2, 5.12]) this amounts to

$$\mathfrak{m}(l(\mathfrak{G}), l^p) = B(l^p).$$

But, again from Lemma 2,

$$\mathfrak{m}(l(\mathfrak{G}), l^p) = \Sigma\{\mathfrak{m}(l^q, l^p); \quad q \in \mathfrak{G}\}.$$

Now, if $p \notin \mathfrak{G}$, from Lemma 3 it follows that $\mathfrak{m}(l^q, l^p) \subseteq C(l^p)$ for all $q \in \mathfrak{G}$, or $B(l^p) = \mathfrak{m}(l(\mathfrak{G}), l^p) \subseteq C(l^p)$, absurd. Then $p \in \mathfrak{G}$ for all $p \in \mathfrak{F}$, and this means that $\mathfrak{F} \subseteq \mathfrak{G}$. Therefore it was shown that $\mathfrak{F} \subseteq \mathfrak{G}$ *if and* *only if* $\mathfrak{a}(\mathfrak{F}) \subseteq \mathfrak{a}(\mathfrak{G})$. This implies that $\mathfrak{F} \to \mathfrak{a}(\mathfrak{F})$ is one-to-one, and the Proposition is proved.

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