A NOTE ON THE STRUCTURE OF MOORE GROUPS

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1. **Introduction.** A locally compact group G will be called a Moore group if every continuous irreducible unitary representation of G is finite dimensional. Let [Moore] denote the class of all Moore groups, and let [Z] denote the class of all locally compact groups such that $G/Z(G)$ is a compact group, where $Z(G)$ denotes the center of G. S. Grosser and M. Moskowitz introduced the classes [Moore] and [Z], and made considerable progress on unifying and organizing the study of various "compactness conditions" in locally compact groups. (See [2], [3], and [4].) Grosser and Moskowitz have shown that [Z]$\subset$[Moore], [3, Theorem 2.1, p. 369], and C. C. Moore has recently shown that $G\in$[Moore] implies that G is an inverse limit of finite extensions of groups $H_\alpha\in$[Z] (see Theorem 3A below). Other results on Moore groups are obtained below by introducing the notion of Takahashi groups. Let [Tak] denote the class of all locally compact groups G such that the derived group G' has compact closure, and G is maximally almost periodic, i.e., there exists a monomorphism from G into a compact group. The main results can be stated as follows:

THEOREM 1. *A group G satisfies $G\in$* [Moore] *if and only if G contains a characteristic subgroup H such that H has finite index in G and $H\in$* [Tak].

THEOREM 2. *A group G satisfies $G\in$* [Moore] *if and only if G is a semidirect product $G=R^n\times_\phi B$, where $B\in$* [Moore] *has a compact identity component B_e, and B contains a normal subgroup H with finite index such that $R^n\times_\phi H$ is a direct product $R^n\times H$.*

Theorem 2 may be interpreted as a type of generalized Freudenthal-Weil theorem (see Theorem 3C below). Consequences of Theorem 1 are that quotient groups of Takahashi groups are Takahashi groups, and (closed) subgroups of Moore groups are Moore groups. (This behavior is a pleasant contrast to results such as the following:

(1) Closed subgroups of [Z]-groups need not be [Z]-groups.

(2) G/H need not be in [MAP] even when $G\in$[MAP] and H is a closed characteristic subgroup of G.)

It follows that the class [Moore] is stable under subgroups, quotient groups, inverse limits, and finite extensions; hence the class

[Moore] constitutes a very well-behaved common generalization of compact groups and abelian groups. (See Theorem 3A for the appropriate notion of inverse limit.) Theorem 1 also implies that every Takahashi group is a Moore group, and hence rounds out the Takahashi duality theorem [9] by showing that the dual structure is based on the set of all equivalence classes of irreducible unitary representations. The inclusion [Tak]$\subset$[Moore] helps to clarify the relationships between a large number of "compactness" conditions which are summarized in §5.

2. **Definitions and notation.** Notation and terminology are taken from Grosser and Moskowitz for those groups which are discussed in [2], [3], and [4]. Let [MAP] denote the class of locally compact groups which are maximally almost periodic. The classes [MAP], [Moore], [Tak] and [Z] are discussed in the introduction. We will also use the following classes:

[Kur] = Kuranishi groups = locally compact G such that $G\in$ [MAP], and G/G_e is compact, where G_e denotes the identity component of G.

$[FC]^-$ = class of locally compact G such that every conjugacy class has compact closure.

A list of definitions of a large number of related properties is available in §5. (See [4] for a more detailed discussion.)

3. **Background material.**

THEOREM 3A (CHARACTERIZATION OF MOORE GROUPS). *A group G satisfies $G\in$ [Moore] if and only if there is a family $\{K_\alpha\}$ of compact normal subgroups of G such that $\bigcap K_\alpha = e$, and each $G_\alpha = G/K_\alpha$ is a finite extension of a group $H_\alpha \in$ [Z]. In particular, the class [Moore] is stable under finite extensions.*

PROOF. This is as yet an unpublished result of C. C. Moore. Use is made of a nonseparable version of Thoma's Theorem [10].

THEOREM 3B (CHARACTERIZATION OF KURANISHI GROUPS). *A group G satisfies $G\in$ [Kur] if and only if G is a semidirect product $G = R^n \times_\phi K$, where R^n is a vector group, and K is a compact group which contains a subgroup H such that H has finite index in K, and $R^n\times_\phi H$ is a direct product $R^n \times H$.*

PROOF. The structure theorem for the most general $G\in$ [Kur] was obtained by Murakami [8, Theorem 1, p. 120]. (See also [6, Corollary XII.I, p. 56], and [7, Lemma 2, p. 41]). Murakami develops a nice application to the study groups with equal left and right uniformities.

THEOREM 3C (FREUDENTHAL-WEIL). *A connected group G satisfies* $G \in [\mathrm{MAP}]$ *if and only if* $G = R^n \times K$, *the direct product of a vector group and a compact group.*

PROOF [1, Theorem 16.4.6, p. 303].

THEOREM 3D (STRUCTURE THEOREM FOR GROUPS $G \in [\mathrm{FC}]^-$). *G satisfies* $G \in [\mathrm{FC}]^-$ *if and only if there is a compact normal subgroup K such that G is an extension* $e \to K \to G \to V \times D \to e$, *where V is a vector group, and* $D \in [\mathrm{FC}]^-$ *is discrete. It follows that every* $G \in [\mathrm{Tak}]$ *satisfies* $G = R^n \times H$, *where the identity component* H_e *is compact.*

PROOF. To appear. The proof uses [4, Corollary 3.22, p. 50].

THEOREM 3E (STABILITY THEOREMS). *Various results show that an appropriate group G contains an n-dimensional normal vector group if G contains a normal subgroup of the form* $R^n \times H$. *See* [2, *Lemma* 1, *p.* 328], [6, *Theorem* X, *p.* 34], *and* [4, *Theorem* 1.1]. *This existence of stable vector subgroups also applies to many situations where G acts as a group of automorphisms of* $R^n \times H$, *rather than just action by restriction of inner automorphisms.*

4. Proof of Theorems 1 and 2.

PROOF OF THEOREM 1. To establish that $[\mathrm{Tak}] \subset [\mathrm{Moore}]$, start by assuming that G is discrete, and then move on to the case where G is a Lie group with an abelian identity component. The case where G is a Lie group can then be handled by studying the restriction of inner automorphisms to the compact semisimple group $(G_e)'$. (Use [5, §6 and Corollary 6.5, p. 122].) Every $G \in [\mathrm{Tak}]$ satisfies $G \in [\mathrm{SIN}]$. and hence there are arbitrarily small compact normal subgroups with Lie group quotients. Conversely, if $G \in [\mathrm{Moore}]$, then use Theorem 3A and the inclusion $[Z] \subset [\mathrm{FD}]^-$. (See definition 5.2, and also [2, Corollary 1, p. 331].) The subgroup H can be chosen as the union of all conjugacy classes which have compact closure.

PROOF OF THEOREM 2. Use Theorems 1, 3D and 3E to obtain a normal subgroup $R^n \times M$ of finite index, where $M \in [\mathrm{Tak}]$, and R^n is normal in G. The subgroup P of (topologically) periodic elements of $R^n \times M$ must satisfy $P \subset M$, and P is closed by [4, Theorem 3.16]. Moreover, [4, Theorem 3.16] shows that G/P is a finite extension of a torsion free abelian group $A = R^n \times D$ where the dual group $\hat{D}$ is compact connected. Apply Theorem 3E to the action of G on the dual group $(R^n)^\wedge \times D^\wedge$, and thus obtain an isomorphic copy D_1 of D such that $A = R^n \times D_1$ with both factors stable under G. Let H be the inverse image in G of D_1, and then define B by applying Theorem 3B to the Kuranishi group G/H.

5. Definitions and relationships.

5.1. $[FC]^-$ = class of locally compact groups G such that every conjugacy class of G has compact closure.

5.2. $[FD]^-$ = class of locally compact groups G such that the derived group G' has compact closure.

5.3. $[FIA]^-$ = class of locally compact groups G such that the group of inner automorphisms has compact closure in the group Aut$[G]$ of all homeomorphic automorphisms.

5.4. [IN] = class of locally compact groups G such that the identity $e \in G$ is contained in some compact neighborhood which is invariant under all inner automorphisms of G. (This is called the invariant neighborhood property.)

5.5. [Kur] = Kuranishi groups = locally compact [MAP] groups such that G/G_e is compact, where G_e denotes the identity component of G.

5.6. [MAP] = maximally almost periodic groups = locally compact G such that there exists a monomorphism from G into some compact group.

5.7. [Moore] = Moore groups = locally compact G such that every continuous irreducible unitary representation of G is finite dimensional.

5.8. [Mur] = Murakami groups = $[\text{MAP}] \cap [\text{SIN}]$.

5.9. $[P_1]$ = class of all discrete groups.

5.10. $[P_2]$ = class of locally compact G such that the center Z(G) contains a vector group $V = R^n$ such that $G/V \in [P_1]$, that is the identity component G_e is an open vector group (perhaps trivial), and $G_e \subset Z(G)$.

5.11. $[P_3]$ = class of locally compact G such that there exists a compact normal K with $G/K \in [P_2]$.

5.12. $[P_4]$ = class of locally compact G such that there exists a characteristic subgroup H of finite index in G such that $H \in [P_3]$.

5.13. $[P_5]$ = class of all locally compact G such that there exists an open normal H with $H \in [P_3]$.

5.14. [SIN] = subclass of [IN] consisting of those G such that every neighborhood of e contains an invariant neighborhood of e. (This is called the small invariant neighborhoods property.)

5.15. [Tak] = Takahashi groups = [MAP$\cap$[FD]$^-$.

5.16. [UM] = class of locally compact unimodular groups.

5.17. [Z] = class of locally compact G such that $G/Z(G)$ is compact, where $Z(G)$ denotes the center of G.

5.18. *Relationships.* The table below displays relationships between the various properties. (Many of the relationships indicated below are theorems from [2] and [4].) Here an asterisk is to be interpreted as blocking an implication arrow, otherwise for properties X and Y which are horizontally or vertically adjacent, the interpretation is $X \Rightarrow Y$ if Y is either to the right of X or below X. (For instance, [Tak]$\Rightarrow$[FD]$^-$ $\Rightarrow$[FC]$^-$$\Rightarrow$[$P_3$]$\Rightarrow$[$P_4$]$\Rightarrow$[IN]$\Rightarrow$[$P_5$] is a typical string of implications.) An implication of the form [X]$\Rightarrow$[P_j] may be regarded as a structure theorem for groups $G \in$ [X].

			[Tak]	[FIA]$^-$	[SIN]	
			[FD]$^-$	[FC]$^-$	[IN]	
		[Kur]	*	[P_3]	*	[MAP]
[Z]	[Tak]	[Moore]	[Mur]	[P_4]	[IN]	[UM]
			[MAP]	*	[P_5]	

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TWO SIDED IDEALS OF OPERATORS

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1. Let X be a Banach space, and $B(X)$ the Banach algebra of all bounded linear operators in X. The closed two sided ideals of $B(X)$ (actually, of *any* Banach algebra) form a complete lattice $L(X)$. Aside from very concrete cases, $L(X)$ has not yet been determined; for instance, when $X=l^p$, $1\leqq p<\infty$, $L(X)$ is a chain (i.e., totally ordered) with three elements: $\{0\}$, $B(X)$ and the ideal $C(X)$ of compact operators (see [3]). On the other hand, it is known [2, 5.23] that for $X=L^p$, $1<p<\infty$, the lattice $L(X)$ is *not* a chain. A treatment for X a Hilbert space of arbitrary dimension can be found in [4]. We aim to exhibit here a Banach space X such that $L(X)$ is both "long" and "wide." Precisely, we have

PROPOSITION. *There exists a real Banach space X with the properties:*
(i) *X is separable, isometric to its dual X^*, and reflexive;*
(ii) *it is possible to assign a closed two sided ideal $\mathfrak{a}(\mathfrak{F})\subset B(X)$ to each finite set of positive integers $\mathfrak{F}$, in such a way that the mapping $\mathfrak{F}\rightarrow\mathfrak{a}(\mathfrak{F})$ is injective and inclusion preserving in both directions: $\mathfrak{F}\subseteq\mathfrak{G}$ if and only if $\mathfrak{a}(\mathfrak{F})\subseteq\mathfrak{a}(\mathfrak{G})$.*

The example is described below, in §3.

2. In the sequel, all Banach spaces are *real* (the complex case can be dealt with similarly). If X, Y are Banach spaces, $\mathfrak{m}(Y, X)$ denotes the set of operators $T\in B(X)$ that can be factorized through Y, i.e., such that $T=SQ$ for suitable bounded linear operators $Q: X\rightarrow Y$, $S: Y\rightarrow X$. If Y is isomorphic (as a Banach space) to its square $Y\times Y$ ($\times$ means cartesian product), then (see [6, Proposition 1.2] or [2, Theorem 5.13]) $\mathfrak{m}(Y, X)$ is a two sided ideal of $B(X)$. $\mathfrak{a}(Y, X)$ will denote the (uniform) closure of $\mathfrak{m}(Y, X)$; thus, if Y is isomorphic to $Y\times Y$, $\mathfrak{a}(Y, X)$ is a *closed two sided ideal* of $B(X)$.

In all that follows, *subspace* means closed lineal subspace; a sub-