LEES' IMMERSION THEOREM AND THE TRIANGULATION OF MANIFOLDS

BY R. LASHOF[1]

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In [4] Lees proves the following immersion theorem for topological manifolds: Let M, M', Q be topological manifolds, M a compact locally flat submanifold of the open manifold M', with $\dim M' = \dim Q = q$, and $\partial Q = \varnothing$. Write $\mathrm{Im}_{M'}(M, Q)$ for the s.s. complex of M' immersions of M in Q; and write $R(TM'/M, TQ)$ for the s.s. complex of representative germs of TM'/M in TQ. A representative germ is a bundle map of the tangent bundle TM' of M', restricted to a neighborhood of M, into the tangent bundle TQ of Q. Two germs are identified if they agree over a common neighborhood of M.

THEOREM (LEES). *If M has a handle decomposition with all handles of index $<Q$; the differential $d\colon \mathrm{Im}_{M'}(M, Q) \to R(TM'/M, TQ)$ is a homotopy equivalence.*

We show here how to simplify some of the hypotheses of this theorem and give applications to the problem of triangulating topological manifolds.

THEOREM A. *In the following two cases, the assumption that M has a handle decomposition may be dropped in Lees' Immersion Theorem.*

(1) $\dim M < \dim Q$.

(2) $\dim M = \dim Q \geqq 5$, *and Q is a piecewise linear (PL) manifold.*

Of course, if M is a PL-manifold, M has a handle decomposition, and hence Lees' theorem applies.

THEOREM B. *In the following cases, $R(TM'/M, TQ)$ may be taken to be the s.s. complex of ordinary bundle maps of TM', restricted to M, into TQ.*

(1) $\dim M = \dim Q$.

(2) $\dim M < \dim Q$, *M a closed submanifold of M' and M the homotopy type of a locally finite simplicial complex.*

We will say that an R^k-bundle $\mathcal{E}$ over a space dominated by a locally finite simplicial complex K admits a PL-bundle structure, if the pullback of $\mathcal{E}$ over K is the underlying topological bundle of a PL-R^k-

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bundle over K. (This is independent of choice of dominating complex and maps.)

Now let M be closed, dim $M=n\geqq 5$, and M simply connected. Write $M^0=M$-open n-ball.

THEOREM C. *If the tangent bundle of M^0 admits a PL-bundle structure, M admits a PL-manifold structure.*

Finally we have

THEOREM D. *Let M_1, M_2 be closed PL-manifolds;* dim $M_i\geqq 5$, *and M_i simply connected, $i=1, 2$. A homeomorphism $h: M_1\to M_2$ is concordant (or weakly isotopic) to a PL-homeomorphism, if and only if the topological bundle map $dh\oplus 1: T(M_1)\oplus 1\to T(M_2)\oplus 1$ is homotopic through topological bundle maps to a PL-bundle map.*

PROOF OF A. Since the essential trick in proving (1) is also used in proving (2), we only prove the latter.

We will need

LEES' LEMMA. *Let M^n be a topological manifold (without boundary), $n\geqq 5$; if M^n-point admits a PL-manifold structure, M^n admits a PL-manifold structure.*

PROOF OF LEMMA. By the Novikov-Siebenmann theorem [5], [6], the end of M^n-point has a neighborhood PL-equivalent to $\Sigma^{n-1}\times R$, Σ^{n-1} a PL-homotopy sphere. But for $n\geqq 5$, $\Sigma^{n-1}\times R$ is PL-equivalent to $S^{n-1}\times R$. By taking t sufficiently large, $S^{n-1}\times t$ is contained in the interior of a disc neighborhood D^n of the point in M^n. By the Shoenflies theorem [2], $S^{n-1}\times t$ bounds a disc D_1^n in D^n. Thus $(M^n-\text{Int } D_1^n)\cup C(S^{n-1}\times t)$ is a PL-manifold homeomorphic to M^n.

PROOF OF A(2). It will be sufficient to show that if either $\text{Im}_{M'}(M, Q)$ or $R(TM'/M, TQ)$ is nonempty, there is a neighborhood V of M in M' that admits a PL-manifold structure. For then there is a compact PL-manifold N, with $M\subset \text{Int } N\subset V'$, V any sufficiently small neighborhood of M. Since N has a handle decomposition, we may apply Lees' theorem to N, and the result follows easily.

Now if $\text{Im}_{M'}(M, Q)$ is nonempty, there is an immersion $f: V\to Q$, V some open neighborhood of M. But then V admits a PL-manifold structure, since Q does.

If $R(TM'/M, TQ)$ is nonempty, there is a neighborhood U of M and a bundle map $\psi: TU\to TQ$. Cover M by a finite number of coordinate neighborhoods $\{V_i'\}$ and let $\{V_i\}$ be a shrinking of this cover, with $\bar{V}_i\subset V_i'$, $\bar{V}_i$ compact. Let $C_t=\bigcup_{i=1}^{r}\bar{V}_i$. We will prove inductively, that if ϕ_{r-1}: $U_{r-1} \to Q$ is an immersion of a neighborhood of C_{r-1}, such that $d\phi_{r-1}$ is homotopic to $\psi | U_{r-1}$, then there is an immersion ϕ_r: $U_r \to Q$, where U_r is a neighborhood of C_r and $d\phi_r \sim \psi | U_r$. (The result is trivial for $C_1 = \overline{V}_1$.)

Triangulate V_r' sufficiently fine, so that any simplex of V_r' that meets C_{r-1} is contained in U_{r-1}. Now $\overline{V}_r$ is contained in a finite subcomplex K of V_r'. Now by induction over the skeletons, we can immerse a neighborhood W of $C_{r-1} \cup K^{(k)}$, using Lemma 2 of [4], with $n = k$, provided $k < q$. Since Q is PL, W admits a PL-manifold structure. Thus a neighborhood W' of C_r admits a PL-structure except at a finite number of points. Therefore, W' admits a PL-structure by Lee's lemma. But then there is a compact PL-manifold N_r, with $C_r \subset \text{Int } N_r \subset W'$. By applying Lees' theorem to N_r we obtain an immersion ϕ_r of a neighborhood U_r of N_r (and hence of C_r) with $d\phi_r$ homotopic to ψ.

This completes the inductive step, and hence there is an immersion ϕ: $U \to Q$, U a neighborhood of M in M'. Hence U admits a PL-structure. Q.E.D.

Proof of B. If dim M = dim Q, then M has a collar in M', and hence is a deformation retract of a neighborhood U. It follows that any bundle map of TM'/M extends canonically to TM'/U, and any two such are canonically homotopic relative to M. Thus the two definitions of $R(TM'/M, TQ)$ are equivalent.

For dim $M <$ dim Q, the author does not know whether a locally flat submanifold is a neighborhood deformation retract; however, it is true stably. First note that if $\mathcal{E}$ is an R^n-bundle over a space X of the homotopy type of a locally finite simplicial complex, the total space $E(\mathcal{E})$ also has this property; and it follows that the projection p: $E(\mathcal{E}) \to X$ and zero section i: $X \to E(\mathcal{E})$ are homotopy inverses, and X is a deformation retract of $E(\mathcal{E})$.

Now M has a normal bundle ν in $M' \times R^k$, k sufficiently large, and M is a (strong) deformation retract of $E(\nu)$. Since TM' may be lifted to a bundle τ over $M' \times R^k$ such that $\tau | M' \times 0 = TM'$, it follows easily that the two definitions of $R(TM'/M, TQ)$ are equivalent in this case also.

Proof of C. Embed M^n in S^{n+k}, k sufficiently large so that M^n has a normal R^k bundle ν. Now $\nu | D^n \cong D^n \times R^k$. Removing the $\text{Int}(D^n \times D^k)$ form S^{n+k}, we get an embedding of $(M^0, \partial M^0)$ in $(D^{n+k}, \partial D^{n+k})$ with normal bundle $\nu^0 = \nu | M^0$, since every neighborhood of the zero section of an R^k-bundle contains an equivalent R^k-bundle (see [3]). Note that the tangent bundle of $E(\nu^0)$ is trivial, and $E(\nu^0)$ is a locally finite simplicial complex dominating M^0. Pulling TM^0 back over $E(\nu^0)$, we have that $M^0 \times R^{n+k}$ admits a PL-manifold structure as a PL-manifold W with boundary.

The Novikov-Siebenmann relative splitting theorem [1], [5], [6], provides a PL-manifold Q^0 with $\partial Q^0 = S^{n-1}$, and a PL-homeomorphism $h: Q^0 \times R^{n+k} \to W$. Thus h defines a homotopy equivalence of pairs $\phi: (Q^0, \partial Q^0) \to (M^0, \partial M^0)$ such that $TQ^0 \cong \phi^* TM^0$ as PL-bundles (actually stably isomorphic, but $Q^0 \sim (n-2)$ complex, and stably isomorphic implies isomorphic). Let ψ be a homotopy inverse of ϕ; then $\psi: (M^0, \partial M^0) \to (Q^0, \partial Q^0)$ is covered by a bundle map $\psi_*: TM^0 \to TQ^0$ of topological bundles. Let $Q = Q^0 \cup CS^{n-1}$, and $M' = M$-point.

Then Theorem A(2) applies to produce an immersion in Q of a neighborhood U of M^0 in M. Thus U admits a PL-manifold structure, and by Lees' lemma, M admits a PL-structure. Q.E.D.

Proof of D. Let M be the underlying topological manifold of M_2, and identify it with that of M_1 via h. The condition on dh implies by A(2), that $M \times I$ may be immersed in $M_2 \times R$ so that the immersion is PL with respect to the M_1 structure near $M \times 0$ and with respect to the M_2 structure near $M \times 1$. This gives a PL-structure on $M \times I$ which is a concordance between the M_1 and M_2 structures. The result follows easily.

Bibliography

1. M. A. Armstrong, *Lectures on the Hauptvermutung according to Lashof and Rothenberg*, Mimeographed Notes, Institute for Advanced Study, Princeton, N. J., 1968.

2. M. Brown *A proof of the generalized Shoenflies theorem*, Bull. Amer. Math. Soc. **65** (1960), 74–76.

3. N. Kuiper and R. Lashof, *Microbundles and bundles*. I, Invent. Math. **1** (1966), 1–17.

4. J. Lees, *Immersions and surgeries on topological manifolds*, Bull. Amer. Math. Soc. **75** (1969), 529–534.

5. S. P. Novikov, *Topological invariance of rational Pontrjagin classes*, Dokl. Akad. Nauk SSSR **163** (1965), 298–300=Soviet Math. Dokl. **6** (1965), 921–923.

6. L. Siebenmann, *The obstruction to finding a boundary for an open manifold*, Ph.D. Thesis, Princeton University, Princeton, N. J., 1965.

University of Chicago, Chicago, Illinois 60637