

ON PROPERTIES OF SELF RECIPROCAL FUNCTIONS

BY V. V. L. N. RAO

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Following is the notation of Hardy and Titchmarsh [1]. We denote a function as R_μ if it is self reciprocal for Hankel transforms of order μ , so that it is given by the formula

$$(1.1) \quad f(x) = \int_0^\infty J_\mu(xy) f(y) \sqrt{xy} dy,$$

where $J_\mu(x)$ is a Bessel function of order μ . For $\mu = \frac{1}{2}$ and $-\frac{1}{2}$, $f(x)$ is denoted as R_s and R_c respectively.

Brij Mohan [2] has shown that if $f(x)$ is R_μ , and

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \Gamma\left(\frac{1}{4} + \frac{\mu}{2} + \frac{s}{2}\right) \Gamma\left(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2}\right) \theta(s) x^{-s} ds,$$

where

$$(1.2) \quad \theta(s) = \theta(1-s) \quad \text{and} \quad 0 < c < 1,$$

then $P(x)$ is a Kernel transforming R_μ (R_ν) into R_ν (R_μ). As an example of (1.2) Brij Mohan has shown that the function

$$(1.3) \quad x^{\nu+1/2} e^{-x}$$

is a Kernel transforming R_ν into $R_{\nu+1}$. In particular, putting $\nu = \frac{1}{2}$, we find that the Kernel

$$(1.4) \quad x e^{-x},$$

transforms R_s into $R_{3/2}$. Again, I have shown in a previous paper [3] that the Kernel

$$(1.5) \quad \sqrt{x} e^{-x/2},$$

transforms R_1 into R_2 . From (1.4) and (1.5) we find that "A Kernel transforming R_1 into R_2 will have its square transforming R_s into $R_{3/2}$." Again Sneddon [4] has shown that

$$\int_0^\infty e^{-x} x^m \text{Ln}(x) dx = (-1)^m m! \int_0^\infty \frac{d^{n-m}}{dx^{n-m}} (x^n e^{-x}) dx,$$

$\text{Ln}(x)$ being Laguerre polynomial of order n . Putting $m = n$, we obtain that

$$\int_0^{\infty} e^{-x} x^n \operatorname{Ln}(x) dx = (-1)^n n! \int_0^{\infty} x^n e^{-x} dx.$$

On writing x^2 for x , we find that

$$(1.7) \quad \int_0^{\infty} e^{-x^2} x^{2n+1} \operatorname{Ln}(x^2) dx = (-1)^n n! \int_0^{\infty} x^{2n+1} e^{-x^2} dx.$$

Brij Mohan [5] has shown that the function

$$(1.8) \quad e^{-x^2/2} x^{2n+1/2}$$

is R_{2n} ; while Howell [6] has shown that the function

$$(1.9) \quad e^{-x^2/2} x^{1/2} \operatorname{Ln}(x^2),$$

is R_0 .

Hence, from (1.8) and (1.9) we find that the integral on the left-hand side of (1.7) is the product of the functions which are R_0 and R_{2n} . Again, by writing the integrand on the right-hand side of (1.7) in the form

$$(1.10) \quad x^n e^{-x^2/2} n^{n+1} e^{-x^2/2},$$

we find from (1.8) that it is a product of two functions which are $R_{(n-1/2)}$ and $R_{(n+1/2)}$. Applying the results of (1.8), (1.9) and (1.10) to (1.7), we conclude that the integral of a product of two functions which are R_0 and R_{2n} = the integral of a product of two functions which are $R_{n-1/2}$ and $R_{n+1/2}$. This may be compared with a theorem given by me in a previous paper [7].

Again, the integrand on the right-hand side of (1.7) may also be written in the form

$$(1.11) \quad (x^{n+1/2} e^{-x^2/2})^2,$$

which is a square of an R_n function. Hence from (1.8), (1.9), (1.11) and (1.7), we further conclude that the integral of a square of an R_n function = the integral of the product of two functions which are R_0 and R_{2n} . My thanks are due to Dr. Brij Mohan for his constant guidance in my research work.

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BHOLAKPUR, SECUNDERABAD-3, A. P., INDIA.

REPRESENTATION OF NONLINEAR TRANSFORMATIONS ON L^p SPACES

BY V. J. MIZEL

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This note describes integral representations obtained for a class of nonlinear functionals and nonlinear transformations on the spaces $L^p(T)$ ($1 \leq p \leq \infty$) associated with an arbitrary σ -finite measure space (T, Σ, μ) . The class of functionals considered here differs from those considered in [1], [3], [7], [8], [9] and its study is mainly motivated by its close connection with nonlinear integral equations [6].

In the study of nonlinear integral equations there is a fundamental class of nonlinear transformations, called Urysohn operators [6], taking measurable functions to measurable functions and having the form

$$(1) \quad (Ax)(s) = \int_T \phi(s; x(t), t) dt$$

where S, T are Lebesgue measurable subsets of R^n and $\phi: S \times R \times T \rightarrow R$ is a real valued function which is measurable on $S \times T$ for each fixed value of its second argument and continuous on R for almost all arguments in $S \times T$. An important subclass of (1) consists of those Urysohn operators whose range is in $C(S)$ where S is compact. This subclass includes the case in which the kernel ϕ is independent of its first argument, so that the operator reduces to a real valued functional:

$$(2) \quad F(x) = \int_T \phi(x(t), t) dt.$$

Functionals of the form (2) also play an important role in the theory of generalized random processes in probability [5].