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THE CENTRALIZER OF A REGULAR UNIPOTENT ELEMENT IN A SEMISIMPLE ALGEBRAIC GROUP

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The following question was posed by Springer [2]: is the centralizer G_x of a regular unipotent element x in a semisimple algebraic group G abelian? In this paper we shall give an affirmative answer and also find the number of disjoint components of G_x if it is reducible. The problem is easily reduced to the case in which G is simple, which we henceforth assume. As proved by Springer in [2], reducibility occurs only when the type of G and the characteristic p of the base field Φ are related as follows: C_n $(n \geqq 2)$ and D_n $(n \geqq 4)$ with $p=2$ (here B_n is a homomorphic image of C_n and need not be considered); F_4, G_2, E_6, E_7, with $p=2$, 3 and E_8 with $p=2$, 3, 5.

We shall now sketch our development. We recall that an element x of G is regular if its centralizer G_x has dimension equal to the rank, say r, of G, and that an element is unipotent if its eigenvalues are all 1. Relative to a Cartan decomposition of G let U be the maximal

[1] The results are part of the author's Ph.D. thesis at the University of California at Los Angeles. The author wishes to thank Professor Robert Steinberg for his patient guidance during the preparation of this work.

unipotent subgroup corresponding to a system P of positive roots, and for each $\alpha \in P$ let x_α denote a corresponding isomorphism of Φ into U. We write $\pi = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ for the system of simple roots and $ht(\alpha) = \sum k_i$ for the height of the root $\alpha = \sum k_i \alpha_i$. Each element u of U can be written uniquely as $u = \prod x_\alpha(t_\alpha)$ $(t_\alpha \in \Phi)$, with the terms arranged according to any fixed ordering of the roots. Chevalley [1] has proved the following fundamental result.

LEMMA 1. *The isomorphisms x_α can be so chosen that for $\alpha, \beta \in P$, the commutator $(x_\alpha(t), x_\beta(u))$ equals $\prod_{i,j \in Z^+} x_{i\alpha+j\beta}(C_{ij,\alpha\beta} t^i u^j)$. The order of the product can be arbitrarily fixed and the $C_{ij,\alpha\beta}$ are integers which can be explicitly determined.*

A typical regular unipotent element in G is $\prod_{\alpha \in \pi} x_\alpha(1)$, an element of U; the product may be taken in any order. Steinberg in [4] has proved that any two regular unipotent elements are conjugate.

LEMMA 2. *If x is a regular unipotent element of G, and if $x \in U$, then the centralizer G_x of x is just U_x.*

For proof see [2, p. 131].

We can obtain the unique expressions in U for xu and ux, with $u = \prod x_\alpha(t_\alpha)$ as above. The parameter corresponding to each root α will be a polynomial in the t_α's. Solving $xu = ux$ is equivalent to finding solutions to a system of polynomial equations in q variables with one equation per root. Therefore U_x can be viewed as an algebraic set in Φ^q. For the parameter t_α associated with the root α we define the height of t_α, $ht(t_\alpha)$, to be $ht(\alpha)$. Induction arguments and computers were used to solve the above equations. In all cases, we can describe the solution as follows: there are r (the rank of G) free variables. In addition, the unique variable of height 1 may take on any value in $GF(p)$, and for $p = 2$, G of type E_7 or E_8, the unique variable of height 2 also may take on the values 0 and 1. Hence U_x has p disjoint irreducible components except in the last two cases when there are 4 such components. The identity component, U_x^0, of U_x is obtained by setting the nonfree variables mentioned above equal to zero. It follows that U_x/U_x^0 is a cyclic p-group. This is true for the exceptional cases since if $u \in U_x$, $u = x_{\alpha_1}(1) x_{\alpha_2}(1) \prod_\beta x_\beta(t_\beta)$ with α_1, α_2 simple roots, then $u^2 = x_{\alpha_1+\alpha_2}(1) \prod_\beta x_\beta(t_\beta) \notin U_x^0$. It has been proved by Springer [3] that G_x^0 is abelian. Therefore, U_x^0 is abelian. The parameter corresponding to simple roots is 1 for the element x; thus we may take xU_x^0 as a generator for U_x/U_x^0. Every element u in U_x may be expressed as $x^k u_0$ with k an integer and $u_0 \in U_x^0$. It is clear now that U_x is abelian and hence G_x is abelian.

If we let U_i denote the subgroup of U generated by $\{x_\alpha(t) \mid t \in \Phi, ht(\alpha) \geqq i\}$, then as a further measure of the structure of U_x we form the sequence of numbers $h = h_1 \leqq h_2 \leqq \cdots$ at which dim $(U_x^0 \cap U_h)$ decreases, i.e., of heights at which free parameters for U_x^0 occur. The deviations from the case $p = 0$ (when the h's are well known, see, e.g., [3]) are as follows:

C_n. Here the h's are 1, 3, 5, $\cdots$, $2n-1$ if $p = 0$. All but the last are to be increased by 1 if $p = 2$.

D_n. Increase the first $[(n-2)/2]$ h's, i.e., 1, 3, 5, $\cdots$ by 1.

E_6 or G_2. Increase h_1 from 1 to p (which is 2 or 3).

E_7. Increase h_1, h_3 from 1, 7 to 4, 8 if $p = 2$, and h_1 to 3 if $p = 3$.

E_8. Increase h_1, h_2, h_4 from 1, 7, 13 to 4, 8, 14 if $p = 2$; h_1, h_2 to 3, 9 if $p = 3$; h_1 to 5 if $p = 5$.

F_4. Increase h_1, h_3 from 1, 7 to 2, 8 if $p = 2$, and h_1 to 3 if $p = 3$.

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