VECTOR VALUED MULTIPLIERS AND APPLICATIONS

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Communicated by Gian-Carlo Rota, May 6, 1968

Let $x=(x_1, \cdots, x_n)\in R^n$; $\xi=(\xi_1, \cdots, \xi_m)\in R^m$. We define

$$\Xi^p X^2 = \left\{f; f: R^{n+m} \to C, \quad \text{such that} \quad \Xi^p X^2(f) = \left\{\int_{R^m}\left[\int_{R^n} |f(x, \xi)|^2 dx\right]^{p/2} d\xi\right\}^{1/p} < \infty\right\}.$$

We shall call $C_0^\infty(R^{n+m})$ the class of infinitely differentiable functions in R^{n+m} with compact support. For $f\in\Xi^1X^1$ define the Fourier transform of f by

$$\mathfrak{F}(f)(y, \eta) = \int_{R^{n+m}} \exp(2\pi i(x \circ y + \xi \circ \eta))f(x, \xi)dxd\xi,$$

where $x \circ y = \sum_{j=1}^n x_j y_j$.

Similarly we define the anti-Fourier transform

$$\mathfrak{F}^{-1}(f)(y, \eta) = \int_{R^{n+m}} \exp(-2\pi i(x \circ y + \xi \circ \eta))f(x, \xi)dxd\xi.$$

We shall denote by $\chi_E(x, \xi)$ the characteristic function of the set E. Finally for $f\in C_0^\infty(R^{n+m})$ and $g(x, \xi)$ bounded we define

$$T(f) = \mathfrak{F}^{-1}(g\mathfrak{F}(f)).$$

THEOREM 1 (LITTLEWOOD-PALEY). *Let $\Lambda=(\lambda_1(x), \cdots, \lambda_m(x))$ denote an m-vector of real valued functions. For the multi-index $N=(n_1, \cdots, n_m)(n_s=\pm 1, \pm 2, \cdots)$ define*

$$Q_N = \left\{(x, \xi); 2^{n_s} \leqq |\xi_s - \lambda_s(x)| \leqq 2^{n_s+1}; 1 \leqq s \leqq m\right\}.$$

Consider $f\in\Xi^pX^2$, and set $f_N=\mathfrak{F}^{-1}(X_{Q_N}\mathfrak{F}(f))$. Then

$$B_p^{-m}\Xi^p X^2\left(\left\{\sum_N |f_N|^2\right\}^{1/2}\right) \leqq \Xi^p X^2(f) \leqq B_p^m \Xi^p X^2\left(\left\{\sum_N |f_N|^2\right\}^{1/2}\right), \quad \text{for all } p, \quad 1 < p < \infty$$

(B_p depends on p only).

[1] Research partially supported by NSF GP 7041X.

THEOREM 2. *Let Q_N be as in Theorem 1 and assume $g(x, \xi)$ is a bounded measurable function such that*

$$\frac{\partial^m}{\partial \xi_1 \cdots \partial \xi_m} (\chi_{Q_N} g) = \mu_N(x, \xi) \quad \text{is a finite measure.}$$

(The last equality is to be understood in the sense of distributions.) Then for $f \in C_0^\infty(R^{n+m})$ and for $T(f) = \mathfrak{F}^{-1}(g\mathfrak{F}(g))$ we have

$$\Xi^p X^2(Tf) \leqq B_p^m \left[\sup_N \left\{ \sup_{x \in R_n} \int_{R_m} d \mid \mu_N(x, \xi) \mid \right\} \right] \Xi^p X^2(f)$$

for all p, $1 < p < \infty$.

As a consequence of Theorem 2 we obtain:

THEOREM 3. *Let $\mathfrak{L} = \{l_1, \cdots, l_r\}$ be a finite family of affine functionals from R^m into R, and assume $S \subset R^{n+m}$ has the property that*

$$S \cap \{(x_0, \xi), x_0 \text{ fixed}\} = \{\xi; l_j(\xi) \geqq \lambda_j(x_0); 1 \leqq j \leqq r\}.$$

Set $g(x, \xi) = \chi_s(x, \xi)$. Then

$$\Xi^p X^2(Tf) \leqq B_p^{rm} \Xi^p X^2(f).$$

In particular if $m = 1$ and $S \subset R^{n+1}$ is a finite union of disjoint convex sets (say k sets) then

$$\Xi^p X^2(Tf) \leqq B_p k \Xi^p X^2(f); \quad \text{for all } p, \quad 1 < p < \infty.$$

REMARK 1. The result of Theorem 3 is the best possible of its kind. More explicitly, if $S = \{(x, \xi); x \in R^n, \xi \in R;$ such that $|x|^2 + \xi^2 \leqq 1\}$ and $T(f) = \mathfrak{F}^{-1}(\chi_s \mathfrak{F}(f))$ is a bounded operator from $\Xi^p X_1^{q_1} \cdots X_n^{q_n}$ into itself for all p, $1 < p < \infty$; then $q_1 = q_2 = \cdots = q_n = 2$. This result is essentially known and due to C. S. Herz [2, p. 996], who shows that T is not a bounded operator from $L^p(R^{n+1})$ into itself when $p \leqq 2(n+1)/(n+2)$ or $p \geqq 2(n+1)/n$. The proof can be extended to show the above result (see also Theorem 5).

Another application of Theorem 2 is the following theorem

THEOREM 4. *Let $P(x, \xi)$ and $Q(x, \xi)$ be two polynomials in the ξ-variable ($x \in R^n$, $\xi \in R$) of degrees m_1 and m_2 respectively. Assume that $g(x, \xi) = P(x, \xi)/Q(x, \xi)$ is a bounded measurable function.*

$$\Xi^p X^2(Tf) \leqq B_p(m_1 + m_2)\Xi^p X^2(f) \quad \text{for all } p, \quad 1 < p < \infty.$$

REMARK 2. As in the case of Theorem 3, the result of Theorem 4 is the best possible of its kind. In [3] W. Littman, C. McCarthy and the author prove that $g(x, \xi)=(|x|^2-\xi+i)^{-1}$ is not a multiplier in $L^p(R^{n+1})$ for either $p<2(n+1)/(n+2)$ or $p>2(n+1)/n$; once again the main estimate of the proof actually shows that the conclusion of Remark 1 is equally valid here (see also Theorem 5).

Using some basic results of the Riesz theory of interpolation for spaces of mixed norm (see [1]), it is possible to extend the results of Theorems 3 and 4.

Given two Banach spaces B_0 and B_1, we shall denote by $[B_0, B_1]_\alpha$ $(0\leqq\alpha\leqq 1)$ the α-intermediate space of the Riesz interpolation having for end points B_0 and B_1.

Set

$$B_1^{(p)} = X_1^p X_2^2 \cdots X_n^2,$$

and

$$B_{j+1}^{(p)} = [B_j^{(p)}, X_{j+1}^p X_1^2 \cdots X_n^2]_{j/(j+1)}.$$

THEOREM 5. *Let $g(x, \xi)$ be either the characteristic function of a finite union of convex sets (as in Theorem 3) $(x\in R^n, \xi\in R)$ or the bounded ratio of two polynomials in all variables (as in Theorem 4). Then*

(i) $$\|T(f)\|_{B_{n+1}^{(p)}} \leqq B_p\|f\|_{L_q(R^{n+1})} \quad \textit{for } 1 < p \leqq 2,$$

where $(n+1)/q=1/p+n/2$ $(2n/(n+2)<q\leqq 2)$.

(ii) $$\|T(f)\|_{L_q(R^{n+1})} \leqq B_p\|f\|_{B_{n+1}^{(p)}} \quad \textit{for } 2 \leqq p < \infty,$$

and q as before $(2\leqq q<2(n+1)/n)$.

The constant B_p depends on p and T as in Theorems 3 and 4.

The proofs of these results will appear elsewhere.

REFERENCES

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3. W. Littman, C. McCarthy, N. M. Rivière, *The non-existence of L^p-estimates for certain translation-invariant operators*, Studia Math. 30 (1968).

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