ON THE INTERSECTIONS OF CONES AND SUBSPACES[1]

BY A. BEN-ISRAEL AND A. CHARNES

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Introduction. Farkas' theorem [9], basic to the theory of linear inequalities (e.g. [15], [7]) and its applications (e.g. [6]), was extended to linear topological spaces in [11], [8], [3], [4], using the separation of closed convex cones and points outside them. In this note a separation argument is used to prove a theorem on the intersections of cones and subspaces in locally convex spaces, which in the finite dimensional case reduces to Farkas' theorem. This approach is similar to that in [1], [13] and [14].

Notations. Let E be a locally convex real linear topological space, E^* the space of continuous linear functionals on E. For any subset S of E let

cl(S) denote the closure of S, $S^* = \{x^*: x^* \in E^*, x^*(x) \geqq 0, x \in S\}$.

Similarly for a subset S^* of E^* let

$$^*(S^*) = \{x: x \in E,\ x^*(x) \geqq 0,\ x^* \in S^*\}.$$

If $L \subset E$ is a subspace

$$L^* = L^0 = \{x^*: x^* \in E^*,\ x^*(x) = 0,\ x \in L\}$$

and for a subspace L^* of E^*

$$^*(L^*) = {}^0(L^*) = \{x: x \in E, x^*(x) = 0,\ x^* \in L^*\}.$$

THEOREM. *Let E be a locally convex real linear topological space, L a closed linear subspace in E, C a closed convex cone in E. Then*

$$^*(L^0 \cap C^*) = \mathrm{cl}(L + C). \tag{1}$$

PROOF. Clearly $\mathrm{cl}(L+C) \subset {}^*(L^0 \cap C^*)$.

Conversely suppose there is an x_0 such that $x_0 \in {}^*(L^0 \cap C^*)$, $x_0 \notin \mathrm{cl}(L+C)$. The last fact implies that the convex compact set $\{x_0\}$ can be strictly separated from the closed convex set $\mathrm{cl}(L+C)$, e.g. [5, p. 73]. Thus there is a $y^* \in E^*$ such that $y^*(\mathrm{cl}(L+C)) \geqq 0$,

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$y^*(x_0)<0$. From $y^*(\mathrm{cl}(L+C))\geqq 0$ it follows that $y^*\in L^0$ and therefore $y^*\in C^*$. Therefore $y^*\in L^0\cap C^*$ and $y^*(x_0)<0$ contradicts $x_0 \in {}^*(L^0\cap C^*)$. Q.E.D.

COROLLARY 1. *Let E, F be real normed linear spaces, T: $E\to F$, a continuous linear operator such that its conjugate T': $F^*\to E^*$ satisfies*

$$(2) \qquad ({}^0\mathrm{cl}(R(T')))^0 = \mathrm{cl}(R(T')).$$

Let C be a closed convex cone in E and $f\in F$ be such that the equation

$$(3) \qquad Tx = f \qquad \textit{is solvable.}$$

Then there is a sequence $\{x_k\}$ in C such that

$$(4) \qquad \lim_{k\to\infty} Tx_k = f$$

if and only if for any solution x_0 of (3) *and any point x^* in* $\mathrm{cl}(R(T'))$:

$$(5) \qquad x^* \in C^* \quad \textit{implies} \quad x^*(x_0) \geqq 0.$$

PROOF. A sequence $\{x_k\}$ in C for which (4) holds exists if and only if for any solution x_0 of (3)

$$(6) \qquad x_0 \in \mathrm{cl}(N(T) + C) \quad \text{or} \quad x_0 \in {}^*(N(T)^0 \cap C^*) \quad \text{by (1).}$$

From (2) it follows that $N(T)^0=\mathrm{cl}(R(T'))$, e.g. [**17**, p. 226] so that (6) gives: $x_0\in{}^*(\mathrm{cl}(R(T'))\cap C^*)$ which is (5). Q.E.D.

REMARK. (2) holds if E is norm reflexive or if $R(T')$ is finite dimensional, e.g. [**17**, p. 227].

COROLLARY 2. *Let E, F be real Hilbert spaces, T: $E\to F$ a continuous linear operator, T^*: $F\to E$ its adjoint.*

Let C be a closed convex cone in E and $f\in F$ be such that the equation

$$(3) \qquad Tx = f \qquad \textit{is solvable.}$$

Then there is a sequence $\{x_k\}$ in C such that

$$(4) \qquad \lim_{k\to\infty} Tx_k = f$$

if and only if for any point x^* in $\mathrm{cl}(R(T^*))$ $x^*=\lim_{k\to\infty} T^*y_k^*$

$$(7) \qquad x \in C^* \quad \textit{implies} \quad \lim_{n\to\infty} (y_n^*, f) \geqq 0.$$

PROOF. Follows from Corollary 1 by choosing $x_0=T^+f$, where T^+ is the generalized inverse of T, e.g. [**16**], [**2**]. Indeed the conclusion $x^*(x_0)\geqq 0$ of (5) is rewritten as:

$$0 \leqq x^*(x_0) = \lim_n T^* y_n^*(T^+ f),$$
$$= \lim_n (y_n^*, TT^+ f) = \lim_n (y_n^*, f),$$

since $TT^+ = P_{\mathrm{cl}(R(T))}$, e.g. [2] and $f \in R(T)$. Q.E.D.

REMARKS. (a) If E, F are real finite dimensional spaces and $C = C^*$ is the nonnegative orthant in E, then Corollary 2 reduces to Farkas' theorem, e.g. [18], [10]:

(8) $Tx = f,\ x \geqq 0$ *is solvable if and only if*:

(9) $T^T y \geqq 0$ *implies* $(y, f) \geqq 0$.

(b) The finite dimensional statement of Corollary 1 is:

Let C be any closed convex cone in R^n, A be an $m \times n$ matrix and $b \in R(A)$. Then there is a sequence $\{x_k\}$ in C such that

(10) $$\lim_k Ax_k = b$$

if and only if

(11) $A^T y \in C^*$ *implies* $(b, y) \geqq 0$.

(c) In particular if C is a polyhedral convex cone in R^n then $N(A) + C$ is closed and (11) is equivalent to

(12) $Ax = b, \quad x \in C,$

being solvable.

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