INTERIORITY OF A HOLOMORPHIC MAPPING ON THE SET OF ITS EXCEPTIONAL POINTS[1]

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I. **Introduction.** A mapping $f\colon A \to B$ is said to be *interior* (or *open*) if for every open subset $U \subset A$, $f(U)$ is an open subset of B; it is said to be interior at a point $a \in A$ (or *locally interior* at $a \in A$) if for every open subset $U \subset A$ containing a, $f(a)$ is an interior point of $f(U)$. Clearly a mapping is interior if and only if it is locally interior everywhere on its domain of definition.

The result contained in this note is about the local interiority property of a holomorphic mapping on the set of its exceptional points. We shall restrict our attention to holomorphic mappings $f = (f_1(x), \cdots, f_n(x))\colon D \to C^n$ where D is a domain (open connected set) in $\bar{C}^n$. $\bar{C}^n = \bar{C}^1 \times \cdots \times \bar{C}^1$ where $\bar{C}^1$ is the extended plane of each one of the complex variables x_i. f is said to be holomorphic when each one of the functions f_i is holomorphic on D. Let $J(x)$ be the value of the Jacobian of f at $x \in D$.

The set E of exceptional points of f is by definition $E = \{a \in D \mid a$ is not an isolated point of $f^{-1}f(a)\}$.

II. **Result.** We recall that if $a \notin E$, f is interior at a. In fact, if $a \notin E$ and $J(a) \neq 0$, the property follows immediately from the inverse function theorem (f is a local homeomorphism); if $a \notin E$ and $J(a) = 0$, it follows from a theorem of Osgood [1] (f maps finitely-to-one sufficiently small neighborhoods of a onto neighborhoods of $b = f(a)$). Our result pertains to the case $a \in E$:

THEOREM. *Let $f\colon D \to C^n$, $D \subset \bar{C}^n$, be a holomorphic mapping and let E be the set of exceptional points of f, then the subset E_0 in E such that $E_0 = \{x \in E \mid f$ is interior at $x\}$ is either the empty set or a set of isolated points.*

PROOF. If E is empty, f is everywhere interior in D as shown above. If f is degenerate, i.e., $J(x) \equiv 0$, it is not difficult to show that $E = D$ and $E_0 = \{\varnothing\}$.

Let then f be not degenerate and E not empty. H. Cartan [2] proved that E is an analytic set and $E \subseteq W = \{x \in D \mid J(x) = 0\}$. Com-

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plex-dimension$(W)=n-1$ and complex-dim$(W'=f(W))\leqq n-1$. Let $S=\{S_1, \cdots, S_r\}\subset E$ be the set (finite) of irreducible local analytic varieties passing through a given arbitrary point $a\in E$ and let $V=\{V_1, \cdots, V_s\}$ be the set (finite) of irreducible subvarieties in S which are associated with a, meaning that $f(V)=f(a)=b=(b_1, \cdots, b_n)$. Now we consider any one-complex-dimensional analytic plane Π passing through b and not contained in W'. Let

$$\Pi = \{y\in C^n \mid (y_1-b_1)/\alpha_1 = \cdots = (y_n-b_n)/\alpha_n\}$$

where $\alpha_1, \cdots, \alpha_n$ are complex constants, be that plane. Obviously

$$f^{-1}(\Pi) = \{x\in D \mid (f_1(x)-b_1)/\alpha_1 = \cdots = (f_n(x)-b_n)/\alpha_n\}.$$

This is an analytic set, consequently, [3], locally at the given point $a\in E$ it consists of a finite set of irreducible analytic varieties which will be called θ. Clearly, $\theta\supseteq V$ since $f(V)=b$ and $b\in\Pi$.

Case 1. $\theta=V$, then f is not locally interior at a. Indeed, if $N_a\subset D$ is a sufficiently small neighborhood of a, b will be the only point in Π contained in $f(N_a)$; this proves that $b=f(a)$ is on the boundary of (N_a).

Case 2. $\theta\supset V$. This means that $\theta=\{V, \theta_1, \cdots, \theta_p\}$ where $\theta_1, \cdots, \theta_p$ are the irreducible analytic varieties in the local decomposition of $f^{-1}(\Pi)$ which are not contained in V. Since Π is not contained in W', none of the θ_i is contained in W. Hence, each one of the θ_i being mapped under f into Π is itself of complex-dimension 1. This proves that the intersection of the θ_i with E is a set $E^*\subset E$ which consists of isolated points. In order for f to be locally interior at a it is necessary that for every Π defined as above there exist varieties θ_i. Hence, the set $E_0\subset E$ of points where f is locally interior certainly satisfies $E_0\subseteq E^*$ (E^* was defined for a single Π) and therefore E_0 contains at most isolated points. Q.E.D.

As an immediate corollary we obtain a result proved by R. Remmert [4].

COROLLARY. *A holomorphic mapping $f: D\to C^n$, $D\subset \overline{C}^n$, is interior if and only if E is the empty set.*

III. **Examples.** In order to show that the two possibilities for E_0 which were mentioned in the Theorem can actually occur, we give the two following examples.

EXAMPLE 1. $f=(y_1=x_1x_2,\ y_2=x_2)$: $C^2\to C^2$. Here $J(x)=x_2$, $E=W=\{x\in C^2 \mid x_2=0,\ x_1 \text{ arbitrary}\}$, $W'=f(E)=\{0'=(y_1=y_2=0)\}$. It is clear that the set $\Pi-\{0'\}$, where $\Pi=\{y\in C^2 \mid y_2=0,\ y_1 \text{ arbitrary}\}$, is not in the range of f. Thus, $\forall a\in E$ and any open set $N_a\subset C^2\ni a\in N_a$,

$0'=f(a)$ is on the boundary of $f(N_a)$. This proves that $E_0=\{\varnothing\}$.

EXAMPLE 2. $f=(y_1=x_1(x_3-x_1),\ y_2=x_1(x_2+x_3),\ y_3=x_1x_2x_3)$: $C^3\to C^3$. Here $E=\{x\in C^3 \mid x_1=0,\ x_2 \text{ and } x_3 \text{ arbitrary}\}$. We shall show that f is locally interior at $0\in E$, $0=(x_1=x_2=x_3=0)$, by proving that for arbitrarily small $\epsilon_1>0$, $\exists\delta_1(\epsilon_1)>0\ni\forall y\in\text{boundary}(\Sigma')$ and $0<\delta<\delta_1$, $\exists x\in\Sigma\ni f(x)=y$, where Σ and Σ' are open hyperspheres, respectively, centered at 0 and $0'$ with radius ϵ_1 and δ.

From the equations defining f we can derive:

$$x_1^4 - x_1^2(y_2-2y_1)+x_1y_3+y_1(y_1-y_2)=0, \tag{1}$$

$$x_2=(y_2-y_1)/x_1-x_1, \tag{2}$$

$$x_3=y_1/x_1+x_1. \tag{3}$$

Let us consider a surface $\sigma=\{y \mid |y_1|^2+|y_2|^2+|y_3|^2=\epsilon^6\}$ where $0<\epsilon\ll 1$. Our first step is to find a common upper bound for the roots x_1^ν, $\nu=1,\cdots,4$, of equation (1) when $y\in\sigma$. We can write (1) as

$$x_1^2=(y_2-2y_1)/2\pm(y_2^2/4-x_1y_3)^{1/2}. \tag{1'}$$

$\forall y\in\sigma$, we obtain from (1′)

$$|x_1|^2<\frac{3\epsilon^3}{2}+\left(\frac{\epsilon^6}{4}+|x_1|\,\epsilon^3\right)^{1/2}<\frac{3\epsilon^3}{2}+\left(\frac{\epsilon^6}{4}\right)^{1/2}+(|x_1|\,\epsilon^3)^{1/2}$$
$$=2\epsilon^3+(|x_1|\epsilon^3)^{1/2}.$$

Since $\epsilon\ll1$, it is not difficult to see that this inequality holds for

(4) $|x_1|<\epsilon+o(\epsilon^2)$ where $o(\epsilon^2)$ is of the order of ϵ^2 when $\epsilon\to0$.

Now let x_1^m be one of the four roots x_1^ν whose absolute value is larger or equal to the absolute value of all the others. We want to find a lower bound for x_1^m. To that purpose we introduced the following symmetric functions of the x_1^ν, obtained from (1):

$$s_4=x_1^1x_1^2x_1^3x_1^4=y_1(y_1-y_2),$$
$$s_3=x_1^1x_1^2x_1^3+\cdots+x_1^2x_1^3x_1^4=(\text{total of 4 terms})=-y_3,$$
$$s_2=x_1^1x_1^2+\cdots+x_1^3x_1^4=(\text{total of 6 terms})=y_2-2y_1.$$

Clearly:

$$|x_1^m|\geqq|s_4|^{1/4}\geqq|y_1|^{1/4}\big|\,|y_1|-|y_2|\,\big|^{1/4},$$
$$|x_1^m|\geqq|s_3/4|^{1/3}=|y_3/4|^{1/3},$$
$$|x_1^m|\geqq|s_2/6|^{1/2}\geqq|(|y_2|-2|y_1|)/6|^{1/2}.$$

Therefore

$$|x_1^m| \geqq \frac{|y_1|^{1/4}||y_1|-|y_2||^{1/4}+|y_3/4|^{1/3}+|(|y_2|-2|y_1|)/6|^{1/2}}{3}.$$

$\forall y \in \sigma$, it follows from this last inequality that $|x_1^m| > \epsilon^{3/2}/9$. Hence, recalling (4), we have

$$\epsilon^{3/2}/9 < |x_1^m| < \epsilon + o(\epsilon^2). \tag{5}$$

Finally from (2), (3) and using (5) we obtain:

$$|x_2^m| \leqq \frac{|y_2| + |y_1| + |x_1^m|^2}{|x_1^m|} < \frac{2\epsilon^3 + \epsilon^2 + o(\epsilon^3)}{\epsilon^{3/2}/9} = 9\epsilon^{1/2} + o(\epsilon^{3/2}),$$

$$|x_3^m| \leqq \frac{|y_1| + |x_1^m|^2}{|x_1^m|} < \frac{\epsilon^3 + \epsilon^2 + o(\epsilon^3)}{\epsilon^{3/2}/9} = 9\epsilon^{1/2} + o(\epsilon^{3/2}).$$

If ϵ is taken to be sufficiently small, then certainly

$$|x_1^m| < \epsilon + o(\epsilon^2) < 10\epsilon^{1/2}, \qquad |x_2^m| < 9\epsilon^{1/2} + o(\epsilon^{3/2}) < 10\epsilon^{1/2},$$

$$|x_3^m| < 9\epsilon^{1/2} + o(\epsilon^{3/2}) < 10\epsilon^{1/2}.$$

In order to complete the required proof it is enough to put

$$\epsilon_1 = 10\epsilon^{1/2} \quad \text{and} \quad \delta_1 = \epsilon^3 = 10^{-6} \times \epsilon_1^6.$$

By using arguments similar to those given in the proof of the Theorem, it is possible to show that $\forall a \in E$ and $a \neq 0$, f is not interior at a. Thus $E_0 = \{0\} \neq \{\varnothing\}$.

References

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4. R. Remmert *Holomorphe und meromorphe Abbildungen komplexer Räume*, Math. Ann. 133 (1957), 328–370.

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