RELATED PROBLEMS IN PARTIAL DIFFERENTIAL EQUATIONS

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1. **Introduction.** Let $x=(x_1, \cdots, x_n)$ and $D=(D_1, \cdots, D_n)$ where $D_i\phi(x)=\partial\phi(x)/\partial x_i$. Let $D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}$ and let $P(x, D) = \sum_{\alpha; 0\leq|\alpha|\leq m} a_\alpha(x)D^\alpha$ where $|\alpha|=\alpha_1+\cdots+\alpha_n$ and the $a_\alpha(x)$ are given functions of x. Finally, let $S(x)=0$ denote a cylindrical surface in (x, t) space and $B(x, D)$ a nontangential boundary operator whose domain is the manifold $S(x)=0$. The smoothness required of $S(x)=0$ will depend upon the operator $B(x, D)$. We will be concerned with the following pair of initial-boundary value problems:

$$\mathrm{P}_1\begin{cases}\partial u(x,t)/\partial t = P(x,D)u(x,t), & t>0,\\ u(x,0) = \phi(x), & \\ B(x,D)u(x,t) = f(x,t), & x\in S, t>0,\end{cases}$$

and

$$\mathrm{P}_2\begin{cases}\partial^2 v(x,t)/\partial t^2 = P(x,D)v(x,t), & t>0,\\ v(x,0) = 0, \quad v_t(x,0) = \phi(x), & \\ B(x,D)v(x,t) = g(x,t), & x\in S, t>0.\end{cases}$$

We assume that $B(x, D)\phi(x)$ vanishes on $S(x)=0$ and that $P(x, D)\phi(x)$ is continuous.

The interest in this paper will be in relating the solvability of P_2 to P_1 and conversely by means of the Laplace transform and the inverse Laplace transform. The use of the Laplace transform will necessarily impose restrictions on the choices of the functions $f(x, t)$ and $g(x, t)$, but these conditions are satisfied in a wide class of applications. By the symbolism $\mathfrak{L}_s^{-1}\{\psi(x, s)\}_{s\to t^2}$ we understand the inverse Laplace transform with the variable s in the transform and the variable t^2 in the inverted function. We then have the following results:

THEOREM 1. *If* P_1 *is solvable with solution* $u(x, t)$ *and if*

$$g(x,t) = \Gamma(3/2)\mathfrak{L}_s^{-1}\{s^{-3/2}f(x, 1/4s)\}_{s\to t^2}, \tag{1.1}$$

then P_2 *is also solvable and*

$$v(x,t) = \Gamma(3/2)\mathfrak{L}_s^{-1}\{s^{-3/2}u(x, 1/4s)\}_{s\to t^2} \tag{1.2}$$

provided the inverse Laplace transform exists in (1.1) *and* (1.2).

THEOREM 2. *If* P_2 *is solvable with solution* $v(x, t)$ *and if*

$$f(x, t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} \int_0^\infty \xi e^{-\xi^2/4t} g(x, \xi)\, d\xi, \tag{1.3}$$

then P_1 *is solvable and*

$$u(x, t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} \int_0^\infty \xi e^{-\xi^2/4t} v(x, \xi)\, d\xi \tag{1.4}$$

provided the integrals exist in (1.3) *and* (1.4) *for* $t>0$.

The cases of P_1 and P_2 that are usually of interest are those in which $P(x, D)$ is an elliptic operator having a positive definite form. Then the equations in P_1 and P_2 are, respectively, parabolic and hyperbolic. Although the initial value problem in P_2 is not well posed if $P(x, D)$ is an elliptic operator having a negative definite form, certain boundary problems related to this operator conveniently fit into our description. From the standpoint of applications, the uses of Theorems 1 and 2 are clear. A problem P_2 (or P_1) that is complicated may be transformed into a more easily solved problem P_1 (or P_2). Applications of these results will, however, be deferred to a later paper.

2. **Proofs of Theorems 1 and 2.** Through transformations of variables and the introduction of the Laplace transform, it will be shown that problems P_1 and P_2 can be reduced to the same problem.

Introduce the change of variables $u(x, t)=u^*(x, t)+\phi(x)$ and $v(x, t)=v^*(x, t)+t\phi(x)$, respectively, in P_1 and P_2. Then P_1 and P_2 transform, respectively, into the problems

$$P_1^1\begin{cases} u_t^*(x, t) = P(x, D)u^*(x, t) + P(x, D)\phi(x); \qquad u^*(x, 0) = 0, \\ B(x, D)u^*(x, t)\,|_S = f(x, t) \qquad (\text{since } B(x, D)\,\phi(x) = 0 \text{ on } S), \end{cases}$$

and

$$P_2^1\begin{cases} v_{tt}^*(x, t) = P(x, D)v^*(x, t) + tP(x, D)\phi(x), \\ v^*(x, 0) = 0, \qquad v_t^*(x, 0) = 0, \\ B(x, D)v^*(x, t)\,|_S = g(x, t). \end{cases}$$

In P_2^1, introduce the change of variables $t=\tau^{1/2}$. Then P_2^1 becomes

$$P_2^2\begin{cases} 4\tau v_{\tau\tau}^* + 2v_\tau^* = P(x, D)v^*(x, \tau^{1/2}) + \tau^{1/2}P(x, D)\phi(x), \\ v^*(x, 0) = 0, \qquad \lim_{\tau\to 0} v_\tau^*(x, \tau^{1/2}) = 0, \\ B(x, D)v^*(x, \tau^{1/2})\,|_S = g(x, \tau^{1/2}). \end{cases}$$

Now introduce the Laplace transform in P_2^2 by transforming on the variable τ with transformed variable s. Then $\bar{v}^*(x, s)$, the Laplace transform of $v^*(x, \tau^{1/2})$, satisfies the problem

$$P_2^3\begin{cases} 4s^2 \dfrac{\partial}{\partial s} \bar{v}^*(x, s) + 6s\bar{v}^*(x, s) + P(x, D)\bar{v}^*(x, s) + \dfrac{\Gamma(3/2)}{s^{3/2}} P(x, D)\phi(x) = 0, \\ B(x, D)\bar{v}^*(x, s) \,|_S = \bar{g}(x, s), \end{cases}$$

with $\bar{g}(x, s)$ the Laplace transform of $g(x, \tau^{1/2})$. Finally, a multiplication of the equation and conditions in P_2^3 by $s^{3/2}/\Gamma(3/2)$ leads to the problem

$$P_2^4\begin{cases} 4s^2 \dfrac{\partial}{\partial s}\left\{\dfrac{s^{3/2}\bar{v}^*}{\Gamma(3/2)}\right\} + P(x, D)\left\{\dfrac{s^{3/2}\bar{v}^*}{\Gamma(3/2)}\right\} + P(x, D)\phi(x) = 0, \\ B(x, D)\left\{\dfrac{s^{3/2}}{\Gamma(3/2)} \bar{v}^*(x, s)\right\}\Big|_S = \dfrac{s^{3/2}}{\Gamma(3/2)} \bar{g}(x, s). \end{cases}$$

In P_1^1, introduce the change of variables $t = 1/(4s)$ for $s > 0$. Then P_1^1 transforms into the problem

$$P_1^2\begin{cases} 4s^2 \dfrac{\partial}{\partial s} u^*(x, 1/4s) + P(x, D)u^*(x, 1/4s) + P(x, D)\phi(x) = 0, \\ B(x, D)u^*(x, 1/4s) \,|_S = f(x, 1/4s) \end{cases}$$

with $\lim_{s\to\infty} u^*(x, 1/4s) = 0$.

A comparison of P_2^4 and P_1^2 shows that the functions $u^*(x, 1/4s)$ and $s^{3/2}(\bar{v}^*(x, s)/\Gamma(3/2))$ satisfy (i) the same differential equation and (ii) the same boundary conditions provided that

$$\text{(2.1)} \qquad \begin{aligned} &\text{(a)} \quad f(x, 1/4s) = \frac{s^{3/2}}{\Gamma(3/2)} \bar{g}(x, s), \\ &\text{(b)} \quad \lim_{s\to\infty} s^{3/2}\bar{v}^*(x, s) = 0. \end{aligned}$$

The conditions (2.1a) are those covered by the hypotheses (1.2) and (1.4). Imposing these conditions along with (2.1b), we get

$$\text{(2.2)} \qquad \bar{v}^*(x, s) = \Gamma(3/2)s^{-3/2}u^*(x, 1/4s),$$

and the result (1.1) follows by inversion and our definitions of u^* and v^*. The result (1.3) also follows from (2.2). This completes the proof.

The requirement that $\phi(x)$ be such that $P(x, D)\phi(x)$ is continuous is not necessary. By mollifying $\phi(x)$ or interpreting $P(x, D)\phi(x)$ in the sense of distributions, the continuity requirements can be weakened. Also observe that the method of proof only depended upon the linearity and t independence of the operator $P(x, D)$. This permits P to be a quite general operator. Finally, if P_1 and P_2 reduce to initial value problems, a similar argument is applicable. In this case, conditions on the known function $u(x, t)$ (or $v(x, t)$) can be imposed on the unknown function $v(x, t)$ (or $u(x, t)$). With no boundary conditions on u and v, it is no longer necessary to require that $\phi(x)$ satisfy a boundary condition.

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LOCALLY NICE EMBEDDINGS IN CODIMENSION THREE[1]

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1. **Introduction.** Suppose that K is a k-dimensional compactum in the interior of a topological q-manifold Q, $q-k\geqq 3$. Following Hempel and McMillan [3], we say that K is *locally nice* in Q if $Q-K$ is 1-ULC. Similarly, an embedding $f: K\to \operatorname{Int} Q$ is said to be *locally nice* if $Q-f(K)$ is 1-ULC.

In [1] the authors showed that a locally nice embedding of a compact k-dimensional polyhedron K into Int Q, where Q is a PL q-manifold, is ϵ-tame whenever $q\geqq 5$ and $2k+2\leqq q$. In this announcement we outline the proof that the same is true for embeddings in codimension at least three if K is a compact PL manifold. Specifically, our main result is

THEOREM 1. *Suppose that M and Q are PL manifolds of dimensions m and q, respectively, with M compact, $q\geqq 5$, and $q-m\geqq 3$, and $f: M\to \operatorname{Int} Q$ is a locally nice embedding. Then f is ϵ-tame.*

The following two corollaries serve to demonstrate the usefulness of Theorem 1 as applied to some special locally nice embeddings.

COROLLARY 1.1. *Suppose that P is a locally tame $(q-1)$-complex in the PL q-manifold Q, $q\geqq 5$, and M is a compact PL m-manifold in Int Q, $q-m\geqq 3$, such that $M-P$ is locally tame. Then M is ϵ-tame.*

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