A FATOU-TYPE THEOREM FOR HARMONIC FUNCTIONS ON SYMMETRIC SPACES[1]

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1. **Introduction.** The result to be proved in this article is that if u is a bounded harmonic function on a symmetric space X and x_0 any point in X then u has a limit along almost every geodesic in X starting at x_0 (Theorem 2.3). In the case when X is the unit disk with the non-Euclidean metric this result reduces to the classical Fatou theorem (for radial limits). When specialized to this case our proof is quite different from the usual one; in fact it corresponds to transforming the Poisson integral of the unit disk to that of the upper half-plane and using only a homogeneity property of the Poisson kernel. The kernel itself never enters into the proof.

2. **Harmonic functions on symmetric spaces.** Let G be a semisimple connected Lie group with finite center, K a maximal compact subgroup of G and $\mathfrak{g}$ and $\mathfrak{k}$ their respective Lie algebras. Let B denote the Killing form of $\mathfrak{g}$ and $\mathfrak{p}$ the corresponding orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Let Ad denote the adjoint representation of G. As usual we view $\mathfrak{p}$ as the tangent space to the symmetric space $X=G/K$ at the origin $o=\{K\}$ and accordingly give X the G-invariant Riemannian structure induced by the restriction of B to $\mathfrak{p}\times\mathfrak{p}$. Let Δ denote the corresponding Laplace-Beltrami operator.

Fix a maximal abelian subspace $\mathfrak{a}\subset\mathfrak{p}$ and let M denote the centralizer of $\mathfrak{a}$ in K. If λ is a linear function on $\mathfrak{a}$ and $\lambda\neq 0$ let $\mathfrak{g}_\lambda = \{X\in\mathfrak{g} \mid [H, X]=\lambda(H)X \text{ for all } H\in\mathfrak{a}\}$; λ is called a restricted root if $\mathfrak{g}_\lambda\neq 0$. Let $\mathfrak{a}'$ denote the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. Fix a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$, i.e. a connected component of $\mathfrak{a}'$. A restricted root α is called positive (denoted $\alpha>0$) if its values on $\mathfrak{a}^+$ are positive. Let the linear function ρ on $\mathfrak{a}$ be determined by $2\rho=\sum_{\alpha>0}(\dim \mathfrak{g}_\alpha)\alpha$ and denote the subalgebras $\sum_{\alpha>0}\mathfrak{g}_\alpha$ and $\sum_{\alpha>0}\mathfrak{g}_{-\alpha}$ of $\mathfrak{g}$ by $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ respectively. Let N and $\bar{N}$ denote the corresponding analytic subgroups of G.

By a Weyl chamber in $\mathfrak{p}$ we understand a Weyl chamber in some maximal abelian subspace of $\mathfrak{p}$. The *boundary* of X is defined as the set B of all Weyl chambers in the tangent space $\mathfrak{p}$ to X at o; since this boundary is via the map $kM\to\mathrm{Ad}(k)\mathfrak{a}^+$ identified with K/M, which by the Iwasawa decomposition $G=KAN$ equals G/MAN, this defi-

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nition of boundary is equivalent to Furstenberg's [2] (see also [6] and [4]). In particular the group G acts transitively on B as well as on X. The two actions will be denoted $(g, b)\to g(b)$ and $(g, x)\to g\cdot x$ $(g\in G, b\in B, x\in X)$. Let db denote the unique K-invariant measure on B normalized by $\int_B db=1$. Then according to Furstenberg [2], the mapping $f\to u$ where

$$u(g\cdot o) = \int_B f(g(b))db \qquad (g \in G), \tag{1}$$

is a bijection of the set $L^\infty(B)$ of bounded measurable functions on B onto the set of bounded solutions of Laplace's equation $\Delta u=0$ on X. The function u in (1) is called the *Poisson integral* of f.

If $g\in G$ let $k(g)\in K$, $H(g)\in\mathfrak{a}$ be determined by $g=k(g)\exp H(g)n$ $(n\in N)$. Observe that if g^h denotes hgh^{-1} for $h\in G$ then $k(\bar{n}^m)=k(\bar{n})^m$, $H(\bar{n}^m)=H(\bar{n})$ for $\bar{n}\in\overline{N}$, $m\in M$. According to Harish-Chandra [3, Lemma 44], the mapping $\bar{n}\to k(\bar{n})M$ is a bijection of $\overline{N}$ onto a subset of K/M whose complement is of lower dimension and if f is a continuous function on B, then

$$\int_B f(b)db = \int_{\overline{N}} f(k(\bar{n})M)\exp(-2\rho(H(\bar{n})))d\bar{n} \tag{2}$$

for a suitably normalized Haar measure $d\bar{n}$ on $\overline{N}$. If $a\in A$ we have $ak(\bar{n})MAN=k(\bar{n}^a)MAN$ whence

$$a(k(\bar{n})M) = k(\bar{n}^a)M \tag{3}$$

so the action of a on the boundary corresponds to the conjugation $\bar{n}\to\bar{n}^a$ on $\overline{N}$.

Let $E_1, \cdots, E_r$ be a basis of $\bar{\mathfrak{n}}$ such that each E_i lies in some $\mathfrak{g}_{-\alpha}$, say $\mathfrak{g}_{-\alpha_i}$. Since the map exp: $\bar{\mathfrak{n}}\to\overline{N}$ is a bijection we can, for each $H\in\mathfrak{a}^+$, consider the function $\bar{n}\to|\bar{n}|_H$ defined as follows: If $\bar{n}=\exp(\sum_1^r a_iE_i)$ $(a_i\in R)$ we put

$$|\bar{n}|_H=\operatorname*{Max}_{1\leq i\leq r}(|a_i|^{1/\alpha_i(H)})$$

Since

$$\bar{n}^{\exp tH} = \exp\left(\sum_1^r a_i\exp(-\alpha_i(H)t)E_i\right) \tag{4}$$

we have

$$|\bar{n}^{\exp tH}|_H = e^{-t}|\bar{n}|_H \qquad \text{for } \bar{n}\in\overline{N},\quad t\in R,\quad H\in\mathfrak{a}^+. \tag{5}$$

For $r>0$ let $B_{H,r}$ denote the set $\{\bar{n}\in\overline{N}\mid |\bar{n}|_H<r\}$ and let $V_{H,r}$ denote the volume of $B_{H,r}$ (with respect to the Haar measure on $\overline{N}$).

LEMMA 2.1. *Let $f\in L^\infty(B)$ and u the Poisson integral* (1) *of f. Put $F(\bar{n})=f(k(\bar{n})M)$ for $\bar{n}\in\overline{N}$. Fix $\bar{n}_0\in\overline{N}$ and $H\in\mathfrak{a}^+$ and assume*

$$\frac{1}{V_{H,r}}\int_{B_{H,r}} |F(\bar{n}_0\bar{n}) - F(\bar{n}_0)|\,d\bar{n}\to 0 \tag{6}$$

for $r\to 0$. Then

$$\lim_{t\to+\infty} u(k(\bar{n}_0)\exp tH(\cdot o)) = f(k(\bar{n}_0)M).$$

PROOF. By the Iwasawa decomposition we can write $\bar{n}_0=k(\bar{n}_0)\cdot(a_1n_1)^{-1}$ $(a_1\in A,\ n_1\in N)$ so

$$u(k(\bar{n}_0)\exp tH\cdot o) = u(\bar{n}_0a_1n_1\exp tH\cdot o) = u(\bar{n}_0\exp tHa_1n_1^{\exp(-tH)}\cdot o).$$

But $G=A\overline{N}K$ so $n_1^{\exp(-tH)}=a(t)\bar{n}(t)k(t)$, each factor tending to e as $t\to+\infty$. If $H_t\in\mathfrak{a}$ is determined by

$$\exp tH_t = \exp tHa_1a(t)$$

we have

$$u(k(\bar{n}_0)\exp tH\cdot o) = u(\bar{n}_0\bar{n}(t)^{\exp tH_t}\exp tH_t\cdot o).$$

The function $f'(b)=f(\bar{n}_0\bar{n}(t)^{\exp tH_t}(b))$ has Poisson integral $u'(x)=u(\bar{n}_0\bar{n}(t)^{\exp tH_t}\cdot x)$; using (1) on u' and f' with $g=\exp tH_t$ we get from (2) and (3)

$$u(k(\bar{n}_0)\exp tH\cdot o) - f(k(\bar{n}_0)M)$$

$$= \int_{\overline{N}}(F(\bar{n}_0\bar{n}(t)^{\exp tH_t}\bar{n}^{\exp tH_t}) - F(\bar{n}_0))\exp(-2\rho(H(\bar{n})))d\bar{n}$$

so

$$\begin{aligned}&|u(k(\bar{n}_0)\exp tH\cdot o) - f(k(\bar{n}_0)M)|\\ &\leqq \int_{\overline{N}}|F(\bar{n}_0\bar{n}^{\exp tH_t}) - F(\bar{n}_0)|\exp(-2\rho(H(\bar{n}(t)^{-1}\bar{n})))d\bar{n}.\end{aligned} \tag{7}$$

Now if $c>0$ let $\overline{N}_c$ denote the "square"

$$\overline{N}_c = \left\{\exp\left(\sum_1^r a_iE_i\right)\Big|\ |a_i| \leqq c, 1\leqq i\leqq r\right\}.$$

The integral on the right in (7) equals the sum

$$\int_{\bar{N}_c} \left| F(\bar{n}_0\bar{n}^{\exp tH_t}) - F(\bar{n}_0) \right| \exp(-2\rho(H(\bar{n}(t)^{-1}\bar{n})))d\bar{n}$$
$$(8) \qquad + \int_{\bar{N}-\bar{N}_c} \left| F(\bar{n}_0\bar{n}^{\exp tH_t}) - F(\bar{n}_0) \right| \exp(-2\rho(H(\bar{n}(t)^{-1}\bar{n})))d\bar{n}.$$

Since $\rho(H(\bar{n})) \geqq 0$ for all $\bar{n} \in \bar{N}$ ([3, p. 287]) and since the mapping $\bar{n} \to \bar{n}^{\exp H}$ has Jacobian $\exp(-2\rho(H))$ (cf. (4)) we see that

$$\int_{\bar{N}_c} \left| F(\bar{n}_0\bar{n}^{\exp tH_t}) - F(\bar{n}_0) \right| \exp(-2\rho(H(\bar{n}(t)^{-1}\bar{n})))d\bar{n}$$
$$(9) \qquad \leqq \exp(2\rho(tH_t)) \int_{\bar{N}_c^{\exp tH_t}} \left| F(\bar{n}_0\bar{n}) - F(\bar{n}_0) \right| d\bar{n}.$$

Now $\bar{n} \in \bar{N}_c^{\exp tH_t}$ if and only if

$$\bar{n} = \exp\left(\sum a_i e^{-\alpha_i(tH_t)} E_i\right) \quad \text{where } |a_i| \leqq c$$

and $tH_t - tH$ is bounded (for fixed $\bar{n}_0$ and H). It follows that

$$\bar{N}_c^{\exp tH_t} \subset B_{H,de^{-t}} \qquad \text{for all } t \geqq 0,$$

$d = d(H, \bar{n}_0, c)$ being a constant. But since the map $\exp: \bar{\mathfrak{n}} \to \bar{N}$ is measure-preserving it is clear that

$$V_{H,de^{-t}} = \exp(-2\rho(H)t)d_1 \qquad t \geqq 0$$

where $d_1 = d_1(H, \bar{n}_0, c)$ is another constant. Also

$$\exp(2\rho(tH_t)) \leqq \exp(2\rho(tH))d_2$$

where $d_2(H, \bar{n}_0)$ is a constant. Thus the right hand side of (9) can be majorized for all $t \geqq 0$:

$$\exp 2\rho(tH_t) \int_{\bar{N}_c^{\exp tH_t}} \left| F(\bar{n}_0\bar{n}) - F(\bar{n}_0) \right| d\bar{n}$$
$$(10) \qquad \leqq d_3 \frac{1}{V_{H,de^{-t}}} \int_{B_{H,de^{-t}}} \left| F(\bar{n}_0\bar{n}) - F(\bar{n}_0) \right| d\bar{n}$$

where d and d_3 are constants depending on H, $\bar{n}_0$ and c.

On the other hand, if $\| \ \|_\infty$ denotes the uniform norm on $\bar{N}$ the second term in (8) is majorized by

$$2\|F\|_\infty \int_{\bar{N}-\bar{N}_c} \exp(-2\rho(H(\bar{n}(t)^{-1}\bar{n})))d\bar{n}$$
$$(11) \qquad = 2\|F\|_\infty \left(1 - \int_{\bar{n}(t)\bar{N}_c} \exp(-2\rho(H(\bar{n})))d\bar{n}\right).$$

Now given $\epsilon > 0$ we first choose c so large that

$$2\|F\|_\infty\left(1 - \int_{\overline{N}_{c/2}} \exp(-2\rho(H(\bar{n})))d\bar{n}\right) < \epsilon/2;$$

since $\bar{n}(t) \to e$ for $t \to +\infty$ we can choose t_1 such that $\bar{n}(t)\overline{N}_c \supset \overline{N}_{c/2}$ for $t \geqq t_1$. Then the expression in (11) is $< \epsilon/2$ for $t \geqq t_1$; by our assumption (6) we can choose t_2 such that the right hand side of (10) is $< \epsilon/2$ for $t > t_2$. In view of (7) and (8) this proves the lemma.

The next lemma shows that, for a fixed H, the assumption of Lemma 2.1 actually holds for almost all $\bar{n}_0 \in \overline{N}$.

LEMMA 2.2. *Let $F \in L^\infty(\overline{N})$ and fix $H \in \mathfrak{a}^+$. Then*

$$\lim_{r\to 0} \frac{1}{V_{H,r}} \int_{B_{H,r}} |F(\bar{n}_0\bar{n}) - F(\bar{n}_0)|\, d\bar{n} = 0 \tag{12}$$

for almost all $\bar{n}_0 \in \overline{N}$.

The proof of this result is essentially in the literature: In [1] Edwards and Hewitt give all the necessary arguments for the case of a discrete sequence tending to 0 and everything they do remains trivially valid in the case $r \to 0$. The result in the exact form required here was also proved by E. M. Stein independently of [1] (cf. his expository article [6]).

THEOREM 2.3. *Let u be a bounded solution of Laplace's equation $\Delta u = 0$ on the symmetric space X. Then for almost all geodesics $\gamma(t)$ starting at o*

$$\lim_{t\to\infty} u(\gamma(t)) \quad \textit{exists.} \tag{13}$$

PROOF. Let $S^+ = \{H \in \mathfrak{a}^+ \mid B(H, H) = 1\}$. Then the mapping $(kM, H) \to \mathrm{Ad}(k)H$ is a bijection of $(K/M) \times S^+$ onto a subset of the unit sphere S in $\mathfrak{p}$ whose complement has lower dimension. Since $\dim(K/M - k(\overline{N})/M) < \dim K/M$ the mapping $(\bar{n}, H) \to \mathrm{Ad}(k(\bar{n}))H$ is a bijection of $\overline{N} \times S^+$ onto a subset of S whose complement in S has lower dimension. If $\overline{N}_H$ denotes the set of $\bar{n}_0$ for which (12) holds (with $F(\bar{n}) = f(k(\bar{n})M)$) and if $S_0 = \bigcup_{H\in S^+} \mathrm{Ad}(k(\overline{N}_H))H$ it follows from the Fubini theorem that $S - S_0$ is a null set. This concludes the proof.

REMARKS. (i) If f is continuous the limit relation

$$\lim_{t\to+\infty} u(k \exp tH \cdot o) = f(kM) \qquad (H \in \mathfrak{a}^+,\ kM \in K/M)$$

follows immediately from (1), (2) and (3), by use of the dominated convergence theorem. (See also [4, Theorem 18.3.2.]) In particular, u has the same limit along all geodesics from o which lie in the same Weyl chamber in $\mathfrak{p}$.

(ii) In the case when X has rank one ($\dim \mathfrak{a}=1$) A. W. Knapp [5] has proved (13), even under the weaker assumption that $f \in L^1(B)$.

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