CONSTRUCTIONS ON LOW-DIMENSIONAL DIFFERENTIABLE MANIFOLDS

BY VALENTIN POÉNARU[1]

Communicated by WIlliam Browder, May 29, 1967

1. This note contains the statements of three theorems on low-dimensional differentiable manifolds (dimensions 3 and 4). The proofs, which use techniques partly connected to [1], will appear elsewhere.

We denote by $M_n+(\phi_\lambda)$ the manifold obtained by adding the handle of index λ, $D_\lambda\times D_{n-\lambda}$, via the embedding ϕ_λ: $S_{\lambda-1}\times D_{n-\lambda}\to\partial M_n$, to M_n. More generally, we shall use the following notation: If $P_{n-1}\subset S_{n-1}=\partial D_n$ is a bounded submanifold and ψ: $P_{n-1}\to M_n$ an embedding, we denote by $M_n+(\psi)$, the space $M_n\cup D_n$, where every $x\in P_{n-1}$ is identified to $\psi(x)\in M_n$. It is understood that, if $\psi(P_{n-1})\subset\partial M_n$, then $M_n+(\psi)$ is a "usual" differentiable manifold, otherwise a "singular" one (see §2).

THEOREM 1. *Let M_3 be a compact, differentiable, homotopy 3-disk. Then $M_3\times I$ is diffeomorphic to D_4 with handles of index 2 and 3 added*:

$$M_3\times I = D_4+(\overset{1}{\phi_2})+\cdots+(\overset{p}{\phi_2})+(\overset{1}{\phi_3})+\cdots+(\overset{p}{\phi_3}).$$

Hence, one can eliminate the handles of index 1 of $M_3\times I$ (compare with the similar procedure, in higher dimensions [2]).

In fact we obtain Theorem 1 from the slightly stronger:

THEOREM 1′. *If M_3 is a compact, differentiable homotopy 3-disk, there exists an integer $p=p(M_3)$ such that*:

$$(M_3\,\#\,(S_2\times I)\,\#\cdots\#\,(S_2\times I)(p\text{ times}))\times I$$
$$= D_4+(\overset{1}{\phi_2})+\cdots+(\overset{p}{\phi_2}).$$

This together with some immersion theory, implies easily the main result from [1].

The next theorem is the main step in proving Theorem 1′. But in order to state it, we need some preparation.

2. We consider the 3-manifold $T_p=(S_1\times D_2)\,\#\,(S_1\times D_2)\,\#\cdots\#\,(S_1\times D_2)$ (p times) and its double $2T_p=(S_1\times S_2)\,\#\,(S_1\times S_2)\,\#\cdots\#\,(S_1\times S_2)$ (p times). ($\#$ means connected sum.) A family of p 2-by-2 disjoint embeddings ϕ_i: $S_1\to T_p$ ($i=1,\cdots,p$) is called

[1] This research has been supported by NSF Grant GP-6299.

"unknotted," if, after a diffeomorphism of T_p, the images become $(S_1\times o)\cup(S_1\times o)\cup\cdots\cup(S_1\times o)$ (p times, o=center of D_2). Similarly, if $o\in S_2$, is some fixed point, a family of p, 2-by-2 disjoint embeddings ψ_i: $S_1\to 2T_p$ $(i=1,\cdots,p)$ is called "unknotted" if after a diffeomorphism of $2T_p$, the images become $(S_1\times o)\cup\cdots\cup(S_1\times o)$ (p times). (The difference between the two notions is illustrated by the Figures 1a, 1b, where "knotted" embeddings $S_1\to T_1$ are presented, such that the composite embeddings $S_1\to T_1\subset 2T_1$ are unknotted. Figure 1b contradicts, unfortunately, the obvious conjecture suggested by Figure 1a.)

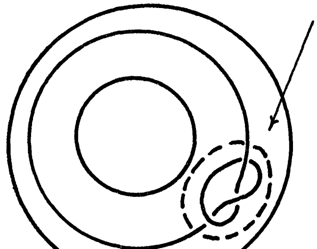

FIGURE 1a

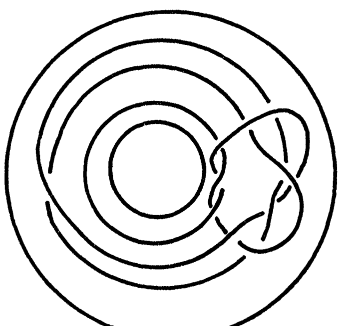

FIGURE 1b

We consider now (compact) 3-manifolds with *singularities*. These will be compact spaces V_3 which are everywhere (bounded) differentiable manifolds, except for a finite number of compact neighborhoods W, which admit descriptions of the following type: We consider two embeddings ϕ, ψ: $I\to S_2=\partial D_3$, such that $\phi(I)\cap\psi(I)$ consists of exactly one point, with transversal intersection, and two thin tubular neighborhoods around them: Φ, Ψ: $I\times I\to S_2=\partial D_3$. $I\times I$ is assimilated to $I\times I\times o\subset\partial(I\times I\times I)=\partial D_3$ and then W is our original D_3 (target of ϕ,ψ) with two other copies of D_3 added along Φ, Ψ:

$$W = D_3 + (\Phi) + (\Psi) = D_3 + (\Psi) + (\Phi).$$

V_3 is "regular" except for a "singular" set $\sigma(V_3)$ which is a bounded 2-manifold, having as connected components various copies of D_2.

We consider resolutions (of singularities) for V_3, Π: $V_3'\to V_3$ where V_3' is a nonsingular 3-manifold, $\Pi^{-1}(x)$ has exactly 2 elements if $x\in\text{int }\sigma(V_3)$ and exactly 1 element if x is regular. (If $x\in\partial\sigma(V_3)$, as we shall see in a moment, $\Pi^{-1}(x)$ has one point in half the cases and two in the other half.) It is moreover understood that, if W is as before, $\Pi^{-1}(W)=W'$ is obtained by cutting the $I\times I\times I$ corresponding to Φ, from $D_3+(\Psi)$ along $\Phi^{-1}(\text{Image } \Phi\cap\text{Image } \Psi)$ (or the $I\times I\times I$ corresponding to Ψ, from $D_3+(\Phi)\cdots$). So $W'=S_1\times D_2$, and passing from W to W', Image $\Phi\cap$Image $\Psi=I\times I$ "blows up" into $\beta=\Phi^{-1}(\text{Image } \Phi\cap\text{Image } \Psi)+\Phi\Phi^{-1}(\text{Image } \Phi\cap\text{Image } \Psi)$, diffeomorphic to $S_1\times I$ (the first summand is in $I\times I\times I$, the other in $D_3+(\Psi)$). We say that Φ (or Ψ) is *specified* in the resolution $\Pi=V_3'\to V_3$. (One remarks that the two $I\times I\times I=D_3$ play a symmetric role in W, but cannot be interchanged with the original D_3; this is easily seen by looking at the sheaf of local homology groups along $\sigma(W)=\text{Image } \Phi\cap\text{Image } \Psi$.)

If Φ is specified in the resolution Π: $V_3'\to V_3$, as above, there exists a canonical embedding j: $W'\to S_3$ which is uniquely determined (up to isotopy) by the requirements that $j(W')$ be unknotted and $j(\beta)$ be contained in a nonsingular 2-disk of S_3.

Let us consider the category $\mathcal{R}$ of resolutions Π: $V_3'\to V_3$ (for all V_3's) where the morphisms are given by commutative squares, having Π on the verticals and embeddings on the horizontals. Let us also consider the category $\mathcal{C}$ consisting of triples (M_4, j, M_3) where M_4 is a bounded differentiable 4-manifold, M_3 a (bounded) differentiable 3-manifold and j: $M_3\to\partial M_4$ an embedding. Morphisms are again commutative squares having the j's on the verticals and embeddings on the horizontals. We have:

LEMMA. *There exists a unique ("thickening") functor Θ: $\mathcal{R}\to\mathcal{C}$ such that, if $\eta\in\mathcal{R}$ is Π: $V_3'\to V_3$ then $\Theta(\eta)=(\Theta_4(\eta), j(\eta), V_3')$ and the following requirements are fulfilled*:

(a) *If V_3 is nonsingular ($\sigma(V_3)=\emptyset$) and η is the (only possible) resolution: identity: $V_3\to V_3$, then $\Theta_4(\eta)=V_3\times I$ and $j(\eta)$ is $V_3\times o\subset\partial(V_3\times I)=V_3\times o+\partial V_3\times I+V_3\times 1$.*

(b) *Θ is compatible with the connected sum # and, more generally, let $V_3={}_oV_3+{}_1V_3$, with ${}_oV_3\cap{}_1V_3=M_2\subset V_3-\sigma(V_3)$ (a compact 2-manifold). If $\eta=(\Pi$: $V_3'\to V_3)$ is a resolution for V_3, M_2 can be lifted to a unique $M_2'\subset V_3'$, and η can be restricted to resolutions ${}_o\eta$ and ${}_1\eta$. By* (a) *and the functoriality of Θ there exist well-defined embeddings $M_2'\times I\subset\partial\Theta_4({}_o\eta)$ and $M_2'\times I\subset\partial\Theta_4({}_1\eta)$ (coming from the corresponding j's). $\Theta_4(\eta)$ is obtained by pasting $\Theta_4({}_o\eta)$ and $\Theta_4({}_1\eta)$ together along $M_2\times I$, and $j(\eta)$ in a similar way from $j({}_o\eta)$, $j({}_1\eta)$.*

(c) *$\Theta(W'\xrightarrow{\Pi}W)=(D_4,\ j\colon W'\to S_3=\partial D_4,\ W')$ where W', W are as above, and j is the canonical embedding. (It is understood that $\Theta(\eta)$ is determined only up to "isomorphism.")*

This lemma is implicit in [1].

3. We are interested in 3-manifolds with singularities V_3, which admit the following description:

We consider T_{2p} and $2p$ differentiable embeddings $\phi^i\colon S_1 \to T_{2p}$ $(i=1, \cdots, 2p)$ such that $\phi^i(S_1)\cap\phi^j(S_1)=\emptyset$ except for $\phi^{2k-1}(S_1)\cap\phi^{2k}(S_1)$ which consists of exactly 2 points, with transversal intersection $(k=1, \cdots, p)$.

We remark that $\phi^{2k-1}(S_1)\cup\phi^{2k}(S_1)$ contains exactly 4 simple circuits of ∂T_{2p} and, for each $k=1, \cdots, p$, we consider a differentiable embedding $\psi^k\colon S \to \partial T_{2p} - \bigcup_1^{2k}\phi^i(S_1)$, "parallel" to one of these 4 circuits. We assume that $\psi^i(S_1)\cap\psi^j(S_1)=\emptyset$. We consider some very thin tubular neighborhoods: Φ^i, $\Psi^j\colon S_1\times I\to\partial T_{2p}$ $(i=1, \cdots, 2p;$ $j=1, \cdots, p)$ of ϕ^i, ψ^j. $S_1\times I$ is assimilated to $(\partial D_2)\times I\subset\partial(D_2\times I)$, and hence we can add $3p$ times $D_2\times I$ along the Φ and Ψ's, to T_{2p}. We get in this way a 3-manifold with singularities

$$V_3 = T_{2p} + (\Phi^1) + \cdots + (\Phi^{2p}) + (\Psi^1) + \cdots + (\Psi^p).$$

We shall consider a resolution $\Pi\colon V_3' \to V_3$ which specifies Φ^2, $\Phi^4, \cdots, \Phi^{2p}$.

We shall also consider, for each $k=1, \cdots, p$ an embedding $\bar{\phi}^{2k-1}\colon S_1\to \operatorname{int} T_{2p}$, very close to ϕ^{2k-1}, and "parallel" to it.

Finally we denote by $\overline{T}_{2p}$ the 3-manifold:

$$\overline{T}_{2p} = T_{2p} + (\partial T_{2p}) \times I \qquad \text{where } \partial T_{2p} = \partial T_{2p} \times o.$$

With this we can state

THEOREM 2. *Let M_3 be a compact homotopy 3-disk. Then, for some $p=p(M_3)$, the differentiable manifold:*

$$\begin{aligned} M_4^p &= (M_3 \# (S_2 \times I) \# \cdots \# (S_2 \times I)(p \text{ times})) \times I \\ &= (M_3 \times I) \# (S_2 \times D_2) \# \cdots \# (S_2 \times D_2)(p \text{ times}) \end{aligned}$$

can be described as follows:

There exists a V_3 as above, for which the following requirement is fulfilled:

(γ) *The $2p$ embeddings $S_1\to 2\overline{T}_{2p}$:*

$$S_1 \xrightarrow{\phi^{2k},\ \bar{\phi}^{2k-1}} T_p \subset \overline{T}_{2p} \subset 2T_{2p} \qquad (k = 1, \cdots, p)$$

are unknotted.

Moreover,

$$\Theta_4(\Pi\colon V_3' \to V_3) = M_4^p \quad (\textit{diffeomorphism}).$$

One remarks that the statement $M_3=D_3$ is equivalent to $M_4^p=D_4 \# (S_2\times D_2) \# \cdots \# (S_1\times D_2)$ (p times) (see [1]). This motivates

THEOREM 3. *Let V_3 be the singular 3-manifold described above, but such that the requirement* (γ) *is replaced by the stronger requirement* (Γ):

(Γ) *The $2p$ embeddings $S_1\to\overline{T}_{2p}$:*

$$S_1 \xrightarrow{\phi^{2k},\ \overline{\phi}^{2k-1}} T_{2p} \subset \overline{T}_{2p} \qquad (k = 1, \cdots, p)$$

are unknotted.

Then:

$$\Theta_4(\Pi: V_3' \to V_3) = D_4 \# (S_2 \times D_2) \# \cdots \# (S_2 \times D_2)(p \text{ times})$$

$$(\textit{diffeomorphism}).$$

BIBLIOGRAPHY

1. V. Poénaru, *Sur les variétés tridimensionnelles ayant le type d'homotopie de S_3*, Séminaire Ehresmann 6 (1963), 1–67.

2. S. Smale, *Generalized Poincaré conjecture in dimensions* >4, Ann. of Math. **74** (1961), 391–406.

NORTHEASTERN UNIVERSITY