A COMBINATORIAL COINCIDENCE PROBLEM

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Let $A \subset E^m$ $(m \geqq 1)$, let $B(o) \subset E^m$ be convex with center of symmetry at o, let n and p be integers $(1 \leqq p \leqq n,\ n \geqq 2)$, and let $f(u)$ be an integrable function defined on A. Let A^n be the Cartesian product of A with itself n times and define $Y \subset A^n$ by

$$Y = \left\{ x = (x_1, \cdots, x_n) : \bigcap_{k=1}^{p} B(x_{i_k}) \neq \emptyset \right.$$
$$\left. \text{for some } i_1, \cdots, i_p,\ 1 \leqq i_1 < \cdots < i_p \leqq n \right\}.$$

The problem of evaluating $J = \int_Y \prod_1^n f(x_i) dx_1 \cdots dx_n$ generalizes a number of questions in probability, queuing theory, scattering, statistical mechanics etc., [1], [2]. Put

$$M = \binom{n}{p},\ S_{i_1 \cdots i_p} = \left\{ (x_1, \cdots, x_n) : \bigcap_{s=1}^{p} B(x_{i_s}) \neq \emptyset \right\}, F(x)$$
$$= \prod_1^n f(x_i),\ dV = dx_1 \cdots dx_n$$

and let the M sets $S_{i_1 \cdots i_p}$ be enumerated as $\{S_k\}$, $k = 1, \cdots, M$. Then by the inclusion-exclusion principle [2]

$$(1) \qquad J = \sum_{r=1}^{n} (-1)^{r+1} \left[\sum_{1 \leqq k_1 < \cdots < k_r \leqq M} \cdots \sum \int_{S_{k_1} \cap \cdots \cap S_{k_r}} F(x) dV \right]$$
$$= \sum_{r=1}^{n} (-1)^{r+1} U_r,$$

say. To help us keep track of different r-tuples of p-tuples, we introduce a generalization of graphs. Let X be a regular simplex in E^{n-1} with the vertices $w_1, \cdots, w_n$, a (d-dimensional) hypergraph G on X is just a collection of some of the (C_{d+1}^n) d-dimensional faces of X; the number of vertices of X lying in G will be denoted by $v(G)$. G is called a (B, r)-hypergraph on X if it consists of r such d-faces and if there are some $v = v(G)$ translates $B_1, \cdots, B_v$ of B such that any $d+1$ of them, say $B_1, \cdots, B_{d+1}$, intersect if the corresponding vertices $w_1, \cdots, w_{d+1}$ lie in a d-face of X included in G.

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G is called connected if no hyperplane in E^{n-1} strictly separates some of its d-faces from the rest of them. Let $t=t(r, d)$ be the number of types of (topologically) distinct (B, r)-hypergraphs on X, let G_j be any one of the jth type, and let $M_{rj}^{d}(n)$ be the number of distinct (B, r)-hypergraphs on X of the jth type. Let $J_0=\int_A f(u)du$, if $d=p-1$ observe that each d-face of a (B, r)-hypergraph corresponds to exactly one set S_k; let

$$J(G) = \int_{S_{k_1}\cap\cdots\cap}\int_{S_{k_r}} F(x)dV$$

where $S_{k_1}, \cdots, S_{k_r}$ are the S-sets corresponding to the d-faces of G. Now we get a formula for the summand U_r of (1):

$$U_r = \sum_{j=1}^{t(r,p-1)} M_{rj}^{p-1}(n)J_0^{n-v(G_j)} \prod_{C(G_j)} J(C(G_j)) \tag{2}$$

where the product is taken over the connected components $C(G_j)$ of G_j. This generalizes some of the so-called cluster expansions of statistical mechanics [3].

In most applications it is found that A and B are simple regular sets (cubes, balls), B is small while A is large, and f is well behaved. (1) and (2) allow us then, in principle at least, to expand J in the powers of a parameter measuring the ratio of sizes of B to A, and to estimate the error of truncation. The integrals $J(C(G_j))$ can rarely be found analytically but the Monte-Carlo method lends itself very well to their numerical evaluation.

The following expansions and identities for iterated binomial coefficients were found in the process of evaluating the numbers $M_{rj}^{p-1}(n)$ in (2). Let $q=q(r, d)$ be the smallest integer $\geqq$ the largest positive root of $r=C_{x,d+1}$, then

$$\left\{\begin{matrix}\binom{n}{d}\\ r\end{matrix}\right\} = \sum_{k=q}^{rd} A_{kr}(d)\binom{n}{k} \tag{3}$$

where

$$A_{kr}(d) = \sum_{j=0}^{k-q}(-1)^j\binom{k}{j}\left\{\begin{matrix}\binom{k-j}{d}\\ r\end{matrix}\right\}. \tag{4}$$

Equating the coefficients of like powers of n in (3) one gets

$$\sum_{j=0}^{dr-q} (-1)^j \binom{dr}{j} \left\{ \begin{matrix} \binom{dr-j}{d} \\ r \end{matrix} \right\} = (dr)!/[r!(d!)^r],$$

$$\sum_{j=0}^{dr-q-1} (-1)^j \binom{dr-1}{j} \left\{ \begin{matrix} \binom{dr-j-1}{d} \\ r \end{matrix} \right\} = d(dr)!(r-1)/2[r!(d!)^r], \text{ etc.}$$

Details of proofs, computations, and applications will appear elsewhere.

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FIXED POINTS OF NONEXPANDING MAPS

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Introduction. This paper is concerned with nonexpanding maps from the unit ball of a real Hilbert space into itself. Browder [1] has established that such maps always possess at least one fixed point. We shall develop a method, which resembles the simple iterative method, for approximating fixed points of such maps. In fact, we shall generate a sequence, $\{x_n\}$, by the recursive formula $x_{n+1} = k_{n+1}f(x_n)$ where f is the map in question and $\{k_n\}$ is a sequence of real numbers. Our main result is Theorem 3 which states sufficient conditions on k_n to insure the strong convergence of x_n to a fixed point of f.

Definitions and preliminary observations. Let H be a Hilbert space with inner product denoted by $(\ ,\ )$ and norm by $\|\ \|$. Let B be the unit ball, $B = \{x \in H \mid \|x\| \leq 1\}$. A map $f: B \to B$ is nonexpanding if $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in B$.

Assume that $f: B \to B$ is nonexpanding. It is not difficult to establish that the set F of fixed points must be convex. Using the con-