THE ADAMS SPECTRAL SEQUENCE FOR $U^*(X, Z_p)$ AND APPLICATIONS TO LIE GROUPS, ETC.

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1. **Preliminaries.** In [1] the structure of the weakly complex *bordism* of 1 connected semisimple Lie groups was studied via the Milnor, Eilenberg-Moore, Rothenberg-Steenrod sequence. See [1] for notation. In this paper we amplify the Adams spectral sequence [2], [3], [4] and relate this tool to the weakly complex cobordism theory. The techniques apply to any finite CW complex. In particular we apply them to real projective spaces and to 1 connected compact semisimple Lie groups.

As in the bordism theory [1], it is useful to introduce coefficients into the cobordism theory. Z_p coefficients arise via [5]. Let

$$\Lambda_p = U^*(pt, Z_p) = Z_p[Y_1, Y_2, \cdots] \dim Y_i = -2i, \qquad i \geqq 1$$

and define $\Lambda_p[1/Y_{p-1}] = \text{direct lim } 1/Y_{p-1}^n\Lambda_p$. $\Lambda_p[1/Y_{p-1}]$ is the ring obtained from Λ_p by making Y_{p-1} a unit. $\Lambda_p[1/Y_{p-1}]$ coefficients can be introduced. $U^*(X, \Lambda_p[1/Y_{p-1}])$ denotes the resulting theory.

The techniques of this paper allow us to extend the theorems in [1]. For example:

THEOREM 1. *Let K be a* 1 *connected compact semisimple Lie group and p a prime. Then $U^*(K, \Lambda_p[1/Y_{p-1}])$ is an exterior algebra over the coefficient ring $\Lambda_p[1/Y_{p-1}]$ generated by rank K elements (except possibly for $U^*(K, \Lambda_2[1/Y_1])$ where K contains E_7 or E_8 as a factor). See* [1, *Theorem* 2].

We intend to make further applications in the detailed version of this paper and remove the "except possibly" statement in the above theorem.

2. **The setting.** Let $\mathfrak{I}$ denote the category of CW complexes having only finitely many cells in each dimension and maps between such spaces. A spectrum X consists of an integer N and spaces $X_i \in \mathfrak{I}$, $i \geqq N$, together with an explicit imbedding $SX_i \to X_{i+1}$. Given two spectra X and Y, a map $f: X \to Y$ is an integer $M \geqq 0$ and maps $f_i: X_i \to Y_i$, $i \geqq M$, commuting with suspensions in the obvious way. A homotopy h between f and g is an integer M' and homotopies h_i

between f_i and g_i, $i \geqq M$. $[X, Y]$ denotes the set of homotopy classes of maps from X to Y. S^rX is the spectrum whose ith space is S^rX_i. $Y^{(r)}$ is the spectrum whose ith space is Y_{i+r}. Define $\Pi_r(X, Y) = [S^rX, Y]$, $r \geqq 0$, and $\Pi_r(X, Y) = [X, Y^{(r)}]$, $r \leqq 0$. This definition was motivated by [3] and enjoys the following properties: (1) For the Eilenberg-MacLean spectrum $\mathcal{K}(p)$, $\mathcal{K}(p)_n = K[Z_p, n]$, $\Pi_*(X, \mathcal{K}(p)) = H^{-*}(W^+, Z_p)$ (reduced cohomology is understood). (2) For the Milnor spectrum M with $M_{2n} = MU(n)$ and $M_{2n+1} = SMU(n)$, $\Pi_*(X, M) = U^{-*}(W)$ is the weakly complex cobordism of W. (Here $W \in \mathfrak{J}$ and $X_i = S^iW^+$.)

Given two spectra X and Y there is a new spectrum $X \wedge Y$ with $(X \wedge Y)_{2n} = X_n \wedge Y_n$ and $(X \wedge Y)_{2n+1} = X_n \wedge Y_{n+1}$. The inclusion of $S(X \wedge Y)_{2n}$ into $(X \wedge Y)_{2n+1}$ involves a sign $(-1)^n$ while the inclusion of $S(X \wedge Y)_{2n-1}$ to $(X \wedge Y)_{2n}$ is the obvious map. Another spectrum of importance is $T_p(W)$ for $W \in \mathfrak{J}$, $T_p(W)$ is constructed as follows: Let $Z_p = S^1 U_p E^2$ be the space obtained from S^1 by attaching a two cell by a map of degree p. For $W \in \mathfrak{J}$, $T_p(W)$ denotes the spectrum whose ith space $T_p(W)_i = S^{i-2} \wedge W \wedge Z_p$, $i \geqq 2$. Here the smash product is taken in $\mathfrak{J}$. Define $U^k(X, Z_p) = \Pi_{-k}(T_p(W^+), M)$ where $W^+ = W \cup w_0$ is the space obtained from W by adding a disjoint base point w_0.

Now suppose that p is an odd prime. Then there is map Δ: $T_p(W^+) \rightarrow T_p(W^+) \wedge T_p(W^+)$. Here is its definition: $Z_p \wedge Z_p$ is homotopically equivalent to $SZ_p \vee S^2Z_p$ so there is a map γ: $S^2Z_p \rightarrow Z_p \wedge Z_p$ which together with the diagonal map d: $W \rightarrow W \wedge W$ produces Δ.

The Whitney sum of two complex vector bundles induces a map $M_i \wedge M_j \rightarrow M_{i+j}$ which in turn provides a map μ: $M \wedge M \rightarrow M$. The two maps Δ and μ determine a product in

$$\Pi_*(T_p(W), M) \text{ via } \Pi_*(T_p(W^+), M) \otimes_{\Pi_*(T(S^0), M)} \Pi_*(T_p(W^+), M) \xrightarrow{\Pi_*(\Delta, \mu)} \Pi_*(T_p(W), M).$$

Δ and μ can be used to introduce a product in the Adams spectral sequence. This product is even defined in the E_1 term of Milnor [4]. These statements are made under the supposition that p was an odd prime. They remain true, however, for $p=2$ but the product is not induced by a map $T_p(W) \rightarrow T_p(W) \wedge T_p(W)$. Another route must be taken. It will be exposed elsewhere.

The most convenient form of the results of the homological analysis of the situation is the following: Let X be a finite CW complex and Q_i be the Milnor cohomology operation [4].

THEOREM 2. *Let p be any prime. There is a spectral sequence of algebras converging to $U^*(X, Z_p)$ whose E_1 term is $\Lambda_p \otimes_{Z_p} H^*(X^+, Z_p)$ (recall reduced cohomology is assumed) as an algebra; moreover, d_1 is the Λ_p morphism defined by $d_1 x = \Sigma_{i=1} Y_{pi-1} Q_i x$ for $x \in H^*(X^+, Z_p)$. This theorem is true for the theory $U^*(X, \Lambda_p[1/Y_{p-1}])$ by replacing Λ_p by $\Lambda_p[1/Y_{p-1}]$.*

Suppose now that X is a finite dimensional H space. The multiplication u can be used to define a coproduct in $U^*(X, \Lambda_p[1/Y_{p-1}])$ and

THEOREM 2′. *The E_1 term of this spectral sequence is $\Lambda_p[1/Y_{p-1}] \otimes_{z_p} H^*(X^+, Z_p)$. The coproduct is $1 \otimes u^*$ and d_1 is a differential of Hopf algebras over $\Lambda_p[1/Y_{p-1}]$. If E_r is free over $\Lambda_p[1/Y_{p-1}]$, then E_r is a Hopf algebra and d_r is a differential of Hopf algebras.*

There is a map of spectra from M to the Eilenberg-MacLane spectrum $K[Z]$ which induces μ_p: $U^*(X, Z_p) \rightarrow H^*(X, Z_p)$. μ_p is a natural transformation of cohomology theories.

COROLLARY 3. *Let $X \in \mathfrak{J}$. If any of the operations Q_i are nonzero in $H^*(X, Z_p)$, μ_p is not onto.*

3. **Applications.** In order to conclude Theorem 1, it suffices to consider the groups SU(n), Sp(n), Spin(n) and the five exceptional groups. Really we must establish the results for pairs (K, p) where K is one of the above list of groups and p is prime. The pairs not included in [1] are $(E_8, 3)$, $(F_4, 3)$, $(E_8, 2)$ and $(E_7, 2)$. Using the knowledge of the cohomology of these groups and tools above we find:

THEOREM 4. *$U^*(E_8, \Lambda_3[1/Y_2]) = \Lambda_3[1/Y_2] \otimes_{Z_3} E(\eta_7, \eta_9, \eta_{19}, \eta_{27}, \eta_{35}, \eta_{37}, \eta_{47}, \eta_{55})$; $U^*(F_4, \Lambda_3[1/Y_2]) = \Lambda_3[1/Y_2] \otimes_{Z_3} E(\eta_7, \eta_{11}, \eta_{15}, \eta_{19})$ as algebras over $\Lambda_3[1/Y_2]$. The subscripts refer to the dimensions of the generators and E is the exterior algebra functor.*

Our tool applies particularly well to Spin(n) and SO(n). The bordism of the first was adequately taken care of in [1]. We deal with SO(n) elsewhere. Let RP^n be real projective n space.

THEOREM 5. *There is a filtration of $U^*(RP^n, Z_2)$ such that $E_0\, U^*(RP^n, Z_2)$ has this description:* (a) *n odd $E_0\, U^*(RP^n, Z_2) = \Lambda_2 \otimes Z_2[W, Y]/I$ where W has* dim 2 *Y has dimension $n-2$ and I is the ideal generated by $W^{n+1/2}$, $W^2 Y$, Y^2 and $\phi_n = \Sigma_{i=1} Y_{2^i-1} W^{2^i}$;* (b) *$n$ even $E_0\, U^*(RP^n, Z_2) = \Lambda_2 \otimes Z_2[W, Y]/I$ where W is of* dim 2, *Y of* dim $n-1$ *and I is the ideal generated by $W^{n+2/2}$, WY, Y^2 and ϕ_n.*

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