D. E. De Giorgi, *Frontiere orientate di misura minima*, Sem. di Mat. de Scuola Norm. Sup. Pisa, 1960–1961, 1–56.

FF. H. Federer, and W. H. Fleming, *Normal and integral currents*, Ann. of Math. 72 (1960), 458–520.

F. W. H. Fleming, *Flat chains over a finite coefficient group*, Trans. Amer. Math. Soc. 121 (1966), 160–186.

M. M. Miranda, *Sul minimo dell'integrale del gradiente di una funzione*, Ann. Scuola Norm. Sup. Pisa (3) 19 (1965), 626–665.

MO. C. B. Morrey, *Multiple integrals in the calculus of variations*, Springer-Verlag, New York, 1966.

R1. E. R. Reifenberg, *Solution of the Plateau problem for m-dimensional surfaces of varying topological type*, Acta Math. 104 (1960), 1–92.

R2. ———, *An epiperimetric inequality related to the analyticity of minimal surfaces*, Ann. of Math. 80 (1964), 1–14.

R3. ———, *On the analyticity of minimal surfaces*, Ann. of Math. 80 (1964), 15–21.

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ZERO-SETS IN POLYDISCS[1]

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For $N=1, 2, 3, \cdots$ the polydisc U^N consists of all $z=(z_1, \cdots, z_N)$ in the space C^N of N complex variables whose coordinates satisfy $|z_j|<1$ for $j=1, \cdots, N$. We write U for U^1. The distinguished boundary of U^N is the torus T^N defined by $|z_j|=1$ $(1\leqq j\leqq N)$. The *zero-set* of a complex function f defined in U^N is the set $Z(f)$ of all $z\in U^N$ at which $f(z)=0$. We call a set $E\subset U^N$ a *zero-set in* U^N if $E=Z(f)$ for some f which is holomorphic in U^N. The main result of this note gives a sufficient condition for zero-sets of bounded functions.

THEOREM 1. *If E is a zero-set in U^N and if no point of T^N is a limit point of E, then there is a bounded holomorphic function F in U^N such that $Z(F)=E$.*

[The term "limit point" refers of course to the topology induced on C^N by the euclidean metric.]

For $N=1$ this is utterly trivial since the hypothesis then forces

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E to be a finite set. For $N>1$, however, the theorem does have content: a qualitative corollary is that zero-sets in U^N which have positive distance from T^N must be rather nice near the rest of the boundary of U^N. More precisely, such sets E must satisfy the following generalized Blaschke condition:

If $\Phi(\lambda)=(\phi_1(\lambda), \cdots, \phi_N(\lambda))$ for $\lambda\in U$, where each ϕ_j is a holomorphic map of U into U, and if

$$Y = \Phi^{-1}(E\cap \Phi(U)) \tag{1}$$

then either $Y=U$ or Y is an at most countable set $\{\lambda_i\}$ such that $\sum(1-|\lambda_i|)<\infty$.

This is a consequence of Blaschke's theorem, applied to the zeros of the bounded function $F\circ\Phi$.

It is also worth noting that the hypothesis of Theorem 1 does not imply the stronger conclusion that F can be chosen so as to be continuous on $U^N\cup T^N$:

THEOREM 2. *There exists a zero-set E in U^2 which has no point of T^2 as a limit point but which has the following property: If F is holomorphic in U^2 and continuous on $U^2\cup T^2$ and if $Z(F)$ contains E, then $F=0$.*

We first sketch the proof of Theorem 2. Let B be a Blaschke product such that every point of the unit circle is a limit point of zeros of B, define $f(z, w)=2w-B(z)$ for $(z, w)\in U^2$, and put $E=Z(f)$. If F is holomorphic in U^2 and continuous on $U^2\cup T^2$ then F has a continuous extension to the closure of U^2, and if $|z|=1$, $F(z, \cdot)$ is holomorphic in U and continuous on $\bar{U}$. Known properties of Blaschke products imply that the closure of E contains all points (z, w) with $|z|=1$, $|w|\leqq\frac{1}{2}$. Hence $F(z, w)=0$ for $|z|=1$, $|w|\leqq 1$ if $E\subset Z(F)$. In particular, $F(z, w)=0$ at every point of T^2, hence $F=0$.

The proof of Theorem 1 starts with a one-variable lemma.

LEMMA 1. *If $0<r<1$, $Q=\{\lambda: r<|\lambda|<1\}$, and*

$$h(\lambda) = \sum_{n=-\infty}^{\infty} a_n\lambda^n, \qquad h_1(\lambda) = \sum_{n=-\infty}^{-1} a_n\lambda^n \tag{2}$$

for $\lambda\in Q$, then

$$\|\mathrm{Re}\, h_1\|_Q \leqq (8/(1-r))\|\mathrm{Re}\, h\|_Q. \tag{3}$$

The norm used in (3) is the supremum over Q.

Suppose $h=u+iv$ in Q and $|u|\leqq 1$. Put $t=\frac{1}{2}(1+r)$. It is easy to see that $|h'(\lambda)|\leqq 4/(1-r)$ if $|\lambda|=t$, so that

$$\left\{\sum_{n=1}^{\infty} |a_n| r^n\right\}^2 \leqq \frac{\pi^2}{6} \cdot \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}$$
$$\leqq \frac{\pi^2}{6} \cdot \sum_{n=-\infty}^{\infty} n^2 |a_n|^2 t^{2n-2}$$
$$= \frac{\pi}{12} \cdot \int_{-\pi}^{\pi} |h'(te^{i\theta})|^2 d\theta < \frac{36}{(1-r)^2}.$$

Hence if $\lambda \in Q$ and $|\lambda|$ is close to r, we have

$$|\operatorname{Re} h_1(\lambda)| = \left| u(\lambda) - \operatorname{Re} a_0 - \operatorname{Re} \sum_{n=1}^{\infty} a_n \lambda^n \right| < 2 + \frac{6}{1-r} < \frac{8}{1-r}.$$

Since $h_1(\lambda) \to 0$ as $\lambda \to \infty$ the lemma now follows from the maximum modulus theorem.

Lemma 1 can be extended to several variables. Let Q^N be the cartesian product of N copies of the annulus Q. Every h holomorphic in Q^N has an absolutely convergent Laurent expansion

$$h(z_1, \cdots, z_N) = \sum a(n_1, \cdots, n_N) z_1^{n_1} \cdots z_N^{n_N} \tag{4}$$

in which the exponents n_i range independently over the set of all integers. For $j = 1, \cdots, N$ let $\pi_j h$ be the series obtained from (4) by replacing $a(n_1, \cdots, n_N)$ by 0 whenever $n_j \geqq 0$.

LEMMA 2. $\|\operatorname{Re} \pi_j h\|_{Q^N} \leqq (8/(1-r)) \|\operatorname{Re} h\|_{Q^N}$.

It suffices to prove this for $j = 1$. Rewrite (4) in the form

$$h(z) = \sum_{n=-\infty}^{\infty} \phi_n(z_2, \cdots, z_N) z_1^n \quad (z \in Q^N) \tag{5}$$

and apply Lemma 1 (regarding $z_2, \cdots, z_N$ as fixed).

We now prove Theorem 1. Fix $r < 1$ so that the distance from E to Q^N is positive. Choose f holomorphic in U^N, so that $Z(f) = E$. Put $z' = (z_2, \cdots, z_N)$. For $k = 0, 1, 2, \cdots$ and $z' \in Q^{N-1}$ put

$$\psi_k(z') = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{(D_1 f)(\zeta, z')}{f(\zeta, z')} \zeta^k d\zeta \tag{6}$$

where D_1 denotes differentiation with respect to the first variable. Each ψ_k is holomorphic in Q^{N-1}. The number of zeros of $f(\cdot, z')$ in U (counted according to multiplicities) is $\psi_0(z')$. So ψ_0 is integer-valued, hence constant, in Q^{N-1}. Call this constant m, let $\alpha_1(z'), \cdots, \alpha_m(z')$ be the zeros of $f(\cdot, z')$, and define

$$(7) \qquad \phi(z) = \prod_{j=1}^{m} (z_1 - \alpha_j(z')) \quad (z \in U \times Q^{N-1}).$$

If $k \geqq 1$, $\psi_k(z') = \sum \alpha_j^k(z')$. The elementary symmetric functions are polynomials in these power sums. It follows that ϕ, f/ϕ and ϕ/f are holomorphic in $U \times Q^{N-1}$. The topological structure of Q^{N-1} therefore shows that there are integers $k_2, \cdots, k_N$ such that $z_2^{k_2} \cdots z_N^{k_N}\phi/f$ has a single-valued continuous logarithm in $U \times Q^{N-1}$. Put $f_1 = z_2^{k_2} \cdots z_N^{k_N}\phi$. Then $f_1 = f \cdot \exp(g_1)$ in $U \times Q^{N-1}$, with g_1 holomorphic, and (7) implies that f_1 and $1/f_1$ are bounded in Q^N.

Similarly, there are holomorphic functions g_j in $Q^{j-1} \times U \times Q^{N-j}$ $(1 \leqq j \leqq N)$ such that, setting

$$(8) \qquad f_j(z) = f(z) \cdot \exp(g_j(z)) \quad (z \in Q^{j-1} \times U \times Q^{N-j}),$$

both f_j and $1/f_j$ are bounded in Q^N.

It follows that f_i/f_j is bounded in Q^N. Hence $\operatorname{Re}(g_i - g_j)$ is bounded in Q^N, for every pair i, j. Also, $\pi_j g_j = 0$, so that

$$(9) \qquad \operatorname{Re} \pi_j g_1 = \operatorname{Re} \pi_j(g_1 - g_j).$$

Lemma 2 (with $h = g_1 - g_j$) now implies that $\operatorname{Re} \pi_j g_1$ is bounded in Q^N, for $j = 1, \cdots, N$. Put

$$(10) \qquad g = (1 - \pi_N) \cdots (1 - \pi_2)(1 - \pi_1)g_1.$$

Since

$$(11) \qquad g_1 - g = \sum \pi_i g_1 - \sum \pi_i\pi_j g_1 + \sum \pi_i\pi_j\pi_k g_1 - \cdots,$$

repeated application of Lemma 2 shows that $\operatorname{Re}(g_1 - g)$ is bounded in Q^N. Since the projections π_j commute with each other, (10) implies that $\pi_j g = 0$ for $1 \leqq j \leqq N$; this says that g extends to a function G holomorphic in U^N.

The function $F = f \cdot \exp(G)$ has the desired properties. For F clearly has the same zeros as f, and in Q^N we have $F = f_1 \cdot \exp(g - g_1)$. Since f_1 and $\operatorname{Re}(g - g_1)$ are bounded in Q^N, F is bounded in Q^N, hence in U^N.

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