NONTRIVIAL $\mathfrak{m}$-INJECTIVE BOOLEAN ALGEBRAS DO NOT EXIST

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We adopt the notation of Sikorski [3] with the following additions. A Boolean algebra $\mathfrak{A}$ is *trivial* iff it has only one element. $\mathfrak{A}$ is $\mathfrak{m}$-*injective* iff $\mathfrak{A}$ is an $\mathfrak{m}$-algebra and whenever we are given $\mathfrak{m}$ algebras $\mathfrak{B}$ and $\mathfrak{C}$ with $\mathfrak{m}$-homomorphisms f, g of $\mathfrak{B}$ into $\mathfrak{A}$ and $\mathfrak{B}$ into $\mathfrak{C}$ respectively, and with g one-one, there is an $\mathfrak{m}$-homomorphism k of $\mathfrak{C}$ into $\mathfrak{A}$ such that $f=k \circ g$ ($\circ$ denotes composition of functions). Obviously every trivial Boolean algebra is $\mathfrak{m}$-injective for any $\mathfrak{m}$. Halmos [1] raised the question concerning what σ-injective Boolean algebras look like, and Linton [2] derived interesting consequences from the assumption that nontrivial σ-injectives exist.

The theorem of the title follows easily from the following two lemmas, the first of which is well known, while the second has some independent interest.

LEMMA 1. *If $\mathfrak{A}$ satisfies the $\mathfrak{m}$-chain condition, $\{A_t\}_{t\in T}$ is a set of elements of $\mathfrak{A}$, and $\bigcup_{t\in T}A_t$ exists, then there is a subset S of T with $\overline{\overline{S}}\leqq \mathfrak{m}$ such that $\bigcup_{s\in S}A_s$ exists and equals $\bigcup_{t\in T}A_t$.*

PROOF. Let $\mathfrak{B}$ be a maximal set of pairwise disjoint elements of $\mathfrak{A}$ such that for every $B\in\mathfrak{B}$ there is a $t\in T$ such that $B\subset A_t$ (such a $\mathfrak{B}$ exists by Zorn's lemma). With every $B\in\mathfrak{B}$ one can associate an element $t(B)$ of T such that $B\subset A_{t(B)}$. By the $\mathfrak{m}$-chain condition, $\overline{\overline{\mathfrak{B}}}\leqq\mathfrak{m}$, and hence also $\{t(B)\}_{B\in\mathfrak{B}}$ is $\mathfrak{m}$-indexed. Now $\bigcup_{B\in\mathfrak{B}}B$ exists and equals $\bigcup_{t\in T}A_t$. For, if this is not true then, by virtue of the fact that $B\subset\bigcup_{t\in T}A_t$ for each $B\in\mathfrak{B}$, it follows that there is a $C\neq\Lambda$ such that $B\cap C=\Lambda$ for all $B\in\mathfrak{B}$, while $C\subset\bigcup_{t\in T}A_t$. Then $C\cap A_{t_0}\neq\Lambda$ for a certain $t_0\in T$, and $\mathfrak{B}\cup\{C\cap A_{t_0}\}$ is a set properly including $\mathfrak{B}$ with all the properties of $\mathfrak{B}$. This contradiction shows that $\bigcup_{B\in\mathfrak{B}}B$ exists and equals $\bigcup_{t\in T}A_t$. Obviously, then, $\bigcup_{B\in\mathfrak{B}}A_{t(B)}$ also exists and equals $\bigcup_{t\in T}A_t$, as desired.

LEMMA 2. *For every $\mathfrak{m}$ there is a complete Boolean algebra $\mathfrak{A}$ such that every nontrivial σ-homomorphic image of $\mathfrak{A}$ has cardinality at least $\mathfrak{m}$.*

PROOF. Let $\mathfrak{B}$ be a free Boolean algebra on $\mathfrak{m}$ generators, and let $\mathfrak{A}$ be a completion of $\mathfrak{B}$. By [3, pp. 72, 156], $\mathfrak{A}$ satisfies the σ-chain condition. Let I be a proper σ-ideal of $\mathfrak{A}$. By Lemma 1, I is principal;

say I is generated by $A \in \mathfrak{A}$. Then $\mathfrak{A}/I$ is isomorphic to $\mathfrak{A}|(-A)$ (see [**3**, pp. 30–31]); moreover, I proper implies that $-A \neq V$. But $\mathfrak{A}$ is homogeneous (see [**3**, pp. 106, 152]), and hence $\mathfrak{A}$ is isomorphic to $\mathfrak{A}/I$. Thus $\mathfrak{A}/I$ has at least $\mathfrak{m}$ elements, as desired.

THEOREM. *Nontrivial* $\mathfrak{m}$-*injective Boolean algebras do not exist.*

PROOF. Suppose that $\mathfrak{A}$ is a nontrivial $\mathfrak{m}$-injective Boolean algebra. Let $\mathfrak{B}$ be the two-element subalgebra of $\mathfrak{A}$, and let $\mathfrak{C}$ be a complete Boolean algebra such that every nontrivial σ-homomorphic image of $\mathfrak{C}$ has power greater than $\overline{\overline{\mathfrak{A}}}$. Let f and g be the natural isomorphisms of $\mathfrak{B}$ and $\mathfrak{A}$ and $\mathfrak{A}$ into $\mathfrak{C}$ respectively. By the $\mathfrak{m}$-injectiveness of $\mathfrak{A}$ we obtain an $\mathfrak{m}$-homomorphism from $\mathfrak{C}$ onto a nontrivial subalgebra of $\mathfrak{A}$, which is impossible.

Linton has remarked to the author that this theorem can be improved to show that the category of $\mathfrak{m}$-algebras does not have a cogenerator.

REFERENCES

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2. F. E. J. Linton, *Injective Boolean σ-algebras*, Arch. Math. **17** (1966), 383–387.

3. R. Sikorski, *Boolean algebras*, 2nd ed., Springer, Berlin, 1964.

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