EQUIVARIANT COHOMOLOGY THEORIES[1]

BY GLEN E. BREDON

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Throughout this note G denotes a discrete group. A G-complex is a CW-complex on which G acts by cellular maps such that the fixed point set of any element of G is a *subcomplex*.

On the category of pairs of G-complexes and equivariant homotopy classes of maps, an *equivariant cohomology theory* is a sequence of contravariant functors $\mathcal{H}^n$ into the category of abelian groups together with natural transformations $\delta^n: \mathcal{H}^n(L, \emptyset) \to \mathcal{H}^{n+1}(K, L)$ such that

(1) $\mathcal{H}^n(K \cup L, L) \xrightarrow{\approx} \mathcal{H}^n(K, K \cap L)$ induced by inclusion,

(2) $\cdots \to \mathcal{H}^n(K, L) \to \mathcal{H}^n(K) \to \mathcal{H}^n(L) \to \mathcal{H}^{n+1}(K, L) \to \cdots$ is exact.

(3) If S is a discrete G-set with orbits S_α then

$$\prod_\alpha i_\alpha^*: \mathcal{H}^n(S) \to \prod_\alpha \mathcal{H}^n(S_\alpha)$$

is an isomorphism, where $i_\alpha: S_\alpha \to S$ is the inclusion. (If S/G is finite then (3) follows from the other axioms.)

One should note that, in a sense, the "building blocks" for the G-complexes are the coset spaces G/H and that the equivariant maps $G/H \to G/K$ are also essential data for building G-complexes. Thus we maintain that the "coefficients" of a theory $\mathcal{H}$ should include the groups $\mathcal{H}^n(G/H)$ together with the induced homomorphisms $\mathcal{H}^n(G/K) \to \mathcal{H}^n(G/H)$. We shall make this more precise.

Let $\mathcal{O}_G$ denote the category whose objects are the coset spaces G/H $(H \subset G)$ and whose morphisms are the equivariant maps. A *coefficient system* is defined to be a contravariant functor from $\mathcal{O}_G$ to Ab (the category of abelian groups). The coefficient systems themselves form a category $\mathcal{C}_G = [\mathcal{O}_G^*, Ab]$ which is an abelian category with projectives and injectives.

The following remark is useful. For G-sets S and T let $E(S, T)$ denote the set of equivariant maps $S \to T$. Also for $H \subset G$ we let $S^H = \{s \in S \mid h(s) = s \text{ for all } h \in H\}$. The assignment $f \to f(H)$ clearly yields a one-one correspondence

$$E(G/H, S) \xrightarrow{\approx} S^H.$$

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Thus it follows immediately that an equivariant map $\phi: G/H \rightarrow G/K$ induces a canonical map $\phi^*: S^K \rightarrow S^H$.

Thus a $Z(G)$-module A induces an element $\boldsymbol{A} \in \mathcal{C}_G$ by $\boldsymbol{A}(G/H) = A^H$ and $\boldsymbol{A}(\phi) = \phi^*$. If (K, L) is a G-complex pair with (cellular) chain complex $C_*(K, L)$ we similarly obtain an element $\boldsymbol{C}_*(K, L) \in \mathcal{C}_G$ where $\boldsymbol{C}_*(K, L)(G/H) = C_*(K^H, L^H)$ and so on. Similarly, the homology groups yield $\boldsymbol{H}_*(K, L) \in \mathcal{C}_G$ defined by $\boldsymbol{H}_*(K, L)\ (G/H) = H_*(K^H, L^H)$. If Y is any G-space with base point $y_0 \in Y^G$ we obtain an element $\varpi_q(Y, y_0) \in \mathcal{C}_G$ defined by $\varpi_q(Y, y_0)(G/H) = \pi_q(Y^H, y_0)$.

If $M \in \mathcal{C}_G$ is arbitrary we define the equivariant "classical" cochain group of the G-complex pair (K, L) by

$$C_G^n(K, L; \boldsymbol{M}) = \operatorname{Hom}(\boldsymbol{C}_n(K, L), \boldsymbol{M})$$

(Hom in $\mathcal{C}_G$) and we define

$$H_G^n(K, L; \boldsymbol{M}) = H^n(C_G^*(K, L; \boldsymbol{M})).$$

These groups are computable, given $\boldsymbol{M}$, the order of difficulty being roughly the same as for ordinary cohomology.

For any equivariant cohomology theory $\mathcal{H}$ we define its "coefficients in degree p" to be the element $\mathcal{H}^p(*) \in \mathcal{C}_G$ defined by

$$\mathcal{H}^p(*)(G/H) = \mathcal{H}^p(G/H).$$

$\mathcal{H}$ is called "classical" if in addition to (1)–(3) above we have the dimension axiom:

(4) $\mathcal{H}^p(*) = 0$ for $p \neq 0$.

Then $\mathcal{H}^0(*)$ denotes the "coefficients" of such a theory. We can prove that, for $\boldsymbol{M} \in \mathcal{C}_G$,

$H_G^*(\cdot\,; \boldsymbol{M})$ *is classical with coefficients* $\boldsymbol{M}$.

Moreover, if $\mathcal{H}$ is classical then there is a functorial isomorphism

$$\mathcal{H}^p(\cdot) \approx H_G^p(\cdot\,; \mathcal{H}^0(*))$$

for finite dimensional G-complexes.

(In particular, the classically defined equivariant cohomology theory with coefficients in the $Z(G)$-module A is, in our notation, $H_G^*(\cdot\,; \boldsymbol{A})$.)

In the case of finite dimensional G-complexes, we can prove more generally that there is a spectral sequence with

$$E_2^{p,q} = H_G^p(K, L; \mathcal{H}^q(*)) \Rightarrow \mathcal{H}^{p+q}(K, L)$$

for any equivariant cohomology theory $\mathcal{H}$.

Another spectral sequence results by applying standard homological algebra to $\mathrm{Hom}(C_*(K, L), M^*)$ where M^* is an injective resolution in $\mathcal{C}_G$ of $M \in \mathcal{C}_G$. This yields the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(H_q(K, L), M) \Rightarrow H_G^{p+q}(K, L; M)$$

where Ext^p refers to the pth right derived functor of Hom in $\mathcal{C}_G$.

We say that a G-space has type (ϖ, n) where $\varpi \in \mathcal{C}_G$ if

$$\begin{aligned} \varpi_q(Y, y_0) &\approx 0 \qquad \text{for } q \neq n, \\ &\approx \varpi \qquad \text{for } q = n \end{aligned}$$

where $y_0 \in Y^G \neq \emptyset$.

We can show, using the existence of projectives in $\mathcal{C}_G$, that G-complexes of type (ϖ, n) exist for all $\varpi \in \mathcal{C}_G$ and all $n \geqq 1$. It is easily proved that if Y has type (ϖ, n) then the equivariant homotopy classes of maps satisfy

$$[K; Y] \approx H_G^n(K; \varpi)$$

for G-complexes K. Thus the $H_G^n(\cdot\,; \tilde{\omega})$ are representable for all $\tilde{\omega} \in \mathcal{C}_G$.

Using this equivariant cohomology theory one can develop an obstruction theory for equivariant extensions of maps. In fact, let Y be a G-space such that Y^H is nonempty, arcwise connected, and n-simple for *each* subgroup $H \subset G$. Let (K, L) be a G-complex pair and let $\phi: K^n \cup L \to Y$ be an equivariant map. Then there is an "obstruction cocycle"

$$c_\phi \in C_G^{n+1}(K, L; \varpi_n(Y))$$

such that $c_\phi = 0$ iff ϕ can be extended equivariantly to $K^{n+1} \cup L$. Moreover, the class

$$[c_\phi] \in H_G^{n+1}(K, L; \varpi_n(Y))$$

is zero iff $\phi \,|\, K^{n-1} \cup L$ can be extended equivariantly to $K^{n+1} \cup L$.

Similarly deformation cochains, primary obstructions, and characteristic classes can be defined in complete analogy with the non-equivariant theory.

The details of this work will be published elsewhere.

UNIVERSITY OF CALIFORNIA, BERKELEY AND
THE INSTITUTE FOR ADVANCED STUDY