AN APPLICATION OF THE CORONA THEOREM TO SOME RINGS OF ENTIRE FUNCTIONS

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1. Introduction. The ring E of entire functions has been extensively investigated in recent years, and a good deal of information on the ideal theory of this ring is now available. The fundamental result here is the theorem of Helmer [3], which asserts that every finitely generated ideal of E is a principal ideal of E. For subrings of E, however, particularly for those determined by growth conditions, this result need no longer hold, and our knowledge of the ideal theory of such rings is quite fragmentary. In this paper we consider some aspects of the ideal theory of certain rings of entire functions defined by growth restrictions on the maximum modulus. For simplicity we restrict the discussion to the ring E_0 of entire functions of exponential type. However, as mentioned below, analogous results hold in more general rings of entire functions. Full details will appear elsewhere.

Recall that an entire function f is of exponential type if there exist constants $A>0$ and $B>0$ such that $|f(z)| \leqq A \cdot \exp(B|z|)$ for all z. It is easy to construct finitely generated ideals of E_0 which are not principal. In fact, there exist functions $f, g \in E_0$ which have no common zeros but which generate a proper ideal of E_0, and here we deal with this latter phenomenon.

MAIN THEOREM. *Let $f_1, \cdots, f_n \in E_0$ and let I denote the ideal of E_0 generated by $f_1, \cdots, f_n$. Then $I=E_0$ if and only if there exist constants $\epsilon>0$ and $A>0$ such that for all z*

$$(*) \qquad |f_1(z)| + \cdots + |f_n(z)| \geqq \epsilon \cdot \exp(-A|z|).$$

This result is quite similar to the Corona Theorem for the Banach algebra H_∞ of all functions bounded and analytic on the unit disc $\{z: |z|<1\}$. (See Hoffman [4] and Carleson [2].) Indeed, our proof of this theorem is based on Carleson's Corona Theorem.

CORONA THEOREM. *Let $f_1, \cdots, f_n \in H_\infty$ and let I be the ideal of H_∞ generated by $f_1, \cdots, f_n$. Then $I=H_\infty$ if and only if there exists a constant $\delta>0$ such that $|f_1(z)| + \cdots + |f_n(z)| \geqq \delta$, $|z|<1$. In this case there exist functions $g_1, \cdots, g_n \in H_\infty$ such that $f_1g_1 + \cdots + f_ng_n = 1$ and such that*

$$\|g_k\|_\infty \leqq \left[\operatorname*{Max}_{1\leqq j\leqq n} \|f_j\|_\infty \right] \cdot K_1 \delta^{-K_2}, \qquad k = 1, \cdots, n,$$

where K_1 and K_2 are constants depending only on the integer n.

Note that this result is invariant under conformal mapping, whence the Corona Theorem holds for the Banach algebra $H_\infty(\Omega)$, Ω being any simply-connected region of the complex plane $\boldsymbol{C}$.

2. **Preliminaries.** In this section we state without proof some results concerning entire functions of exponential type which enable us to apply the Corona Theorem. First, given $f\in E_0$ and $A>0$ we denote the set $\{z\in\boldsymbol{C}\colon |f(z)|<\exp(-A|z|)\}$ by $S(f;A)$. Observe that the components of $S(f;A)$ are simply-connected regions of $\boldsymbol{C}$.

LEMMA. *Let $f\in E_0$ be nonconstant. Then there exists a sequence $\{R_k\}_{k=1}^{\infty}$ of positive constants, with $2R_k\leqq R_{k+1}\leqq 4R_k$ for all $k\geqq 1$, and a constant $A_0>0$ such that $S(f;A)\cap\{|z|=R_k\}=\varnothing$ for all $k\geqq 1$ and all $A\geqq A_0$.*

The proof is omitted, the Lemma being a simple consequence of a minimum modulus theorem for functions of exponential type. (Boas [1, p. 52, Theorem 3.7.4].)

INTERPOLATION THEOREM. *Let $f\in E_0$ be nonconstant and let $A>0$. Let $\{D_\gamma\}$, $\gamma\in\Gamma$ denote the components of $S(f;A)$, and for each $\gamma\in\Gamma$ let g_γ be a function analytic on D_γ. Suppose that $|g_\gamma(z)|\leqq B\cdot\exp(C\cdot|z|)$, $z\in D_\gamma$, where B and C are constants independent of γ. Then there exists $g\in E_0$ such that for each $\gamma\in\Gamma$ the function $(g-g_\gamma)/f$ is analytic on D_γ.*

The proof of this theorem is straightforward, but involves technicalities too lengthy for the present discussion. Note the similarity of this theorem to a result of Carleson ([2, p. 557, Theorem 4]).

3. **Proof of the Main Theorem.** Let $f_1,\cdots,f_n\in E_0$ and let I denote the ideal of E_0 generated by $f_1,\cdots,f_n$. (We suppose that the functions f_j are nonconstant.) First, if $I=E_0$, there exist $g_1,\cdots,g_n\in E_0$ such that $f_1g_1+\cdots+f_ng_n=1$ and the necessity of condition (*) follows immediately.

Conversely, suppose (*) holds. We may assume, without loss of generality, that $\epsilon=2$, so that $|f_1(z)|+\cdots+|f_n(z)|\geqq 2\exp(-A|z|)$ for some $A>0$ and all $z\in\boldsymbol{C}$. Let us apply the above Lemma to the function f_n, and let A_0 and $\{R_k\}_{k=1}^{\infty}$ be the constants for which the conclusions of the Lemma are valid. Let A_1 be a constant with $A_1\geqq\operatorname{Max}\{A, A_0\}$ and with $|f_j(z)|\leqq A_1\exp(A_1|z|)$ for all $z\in\boldsymbol{C}$, $1\leqq j\leqq n$. Then for all $z\in S(f_n;A_1)$ we have $|f_1(z)|+\cdots+|f_{n-1}(z)|\geqq\exp(-A_1|z|)$, and $S(f_n;A_1)\cap\{|z|=R_k\}=\varnothing$ for all $k\geqq 1$.

Now we shall suppose that $S(f_n;A_1)\cap\{|z|\leqq R_1\}=\varnothing$. (The general result follows from this special case, though we shall not justify this statement here.) Let D_γ be a component of $S(f_n; A_1)$. Since $R_k \to +\infty$, there exists $k \geqq 1$ such that $D_\gamma \subset \{R_k < |z| < R_{k+1}\}$. Thus, since $R_{k+1} < 4R_k$, for all $z \in D_\gamma$

$$|f_1(z)| + \cdots + |f_{n-1}(z)| \geqq \exp(-4BR_k).$$

We now apply the Corona Theorem to the functions $f_j | D_\gamma$, $1 \leqq j \leqq n$, considered as elements of the Banach algebra $H_\infty(D_\gamma)$, thereby obtaining functions $g_1^{(\gamma)}, \cdots, g_{n-1}^{(\gamma)} \in H_\infty(D_\gamma)$ such that $f_1 g_1^{(\gamma)} + \cdots + f_{n-1} g_{n-1}^{(\gamma)} = 1$ on D_γ. Furthermore, for $1 \leqq i < n$ we have

$$\sup_{z \in D_\gamma} |g_i^{(\gamma)}(z)| \leqq \operatorname*{Max}_{1 \leqq j < n} \left[\sup_{z \in D_\gamma} |f_j(z)| \right] \cdot K_1 \cdot \exp(4A_1 R_k K_2).$$

However, by our choice of A_1 we have $|f_j(z)| \leqq A_1 \exp(4A_1 R_k)$ for $z \in D_\gamma$, $1 \leqq j < n$. Thus, since $R_k < |z|$ for $z \in D_\gamma$, $|g_i^{(\gamma)}(z)| \leqq B \cdot \exp(C|z|)$ for $1 \leqq i < n$ and $z \in D_\gamma$, where B and C are constants independent of the component D_γ in question.

Now the functions $g_1^{(\gamma)}, \cdots, g_{n-1}^{(\gamma)}$ exist as above for each component D_γ of $S(f_n; A_1)$. Moreover, for each i, $1 \leqq i < n$, the family $\{g_i^{(\gamma)}\}$ satisfies the hypotheses of the Interpolation Theorem, and therefore there exist $g_1, \cdots, g_{n-1} \in E_0$ such that for each component D_γ of $S(f_n; A_1)$ the function $(g_i - g_i^{(\gamma)})/f_n$ is analytic on D_γ, $1 \leqq i < n$. However, $S(f_n; A_1)$ is evidently a neighborhood of the zeros of f_n, and therefore the function

$$g_n = (1 - (f_1 g_1 + \cdots + f_{n-1} g_{n-1}))/f_n$$

is an entire function, hence of exponential type. We have now obtained functions $g_1, \cdots, g_n \in E_0$ with $f_1 g_1 + \cdots + f_n g_n = 1$. That is, $1 \in I$, or $I = E_0$. Q.E.D.

One may obtain bounds on the types of the functions $g_1, \cdots, g_n$. We shall not consider this here, however, as a detailed analysis of the proof of the Interpolation Theorem is required. Note that the Main Theorem follows directly from the Interpolation Theorem in the case $n = 2$, but as yet we have been unable to avoid the use of the Corona Theorem in the general case.

4. Remarks. As mentioned in the Introduction, the methods above have application to other rings of entire functions. Roughly speaking, results analogous to the Main Theorem hold in rings of entire functions for which appropriate versions of the minimum modulus and Interpolation Theorems are valid, since then the Corona Theorem may be applied as above. For example, for each $\rho > 0$ the result corresponding to our Main Theorem holds for the ring of all entire functions of order ρ, finite type, the ring E_0 being the special case $\rho=1$. Using this one then obtains a theorem for the ring of all entire functions of finite order. The details of these theorems, as well as extensions of the Main Theorem to more general rings of entire functions, will be published elsewhere.

Bibliography

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